A remark on the instability of positive solutions to a diffusive logistic equation

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Abstract

We study the stability of positive solution to the boundary value problem

\[-\Delta_p u = \lambda m(x)(u^{\gamma-1} - u^{p-1} - ch(x)), \quad x \in \Omega,\]
\[Bu = 0, \quad x \in \partial\Omega,\]

where \(\Delta_p\) denotes the p-Laplacian operator defined by \(\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z);\) \(p > 1, \gamma(>p), \Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary \(Bu(x) = \alpha g(x)u + (1 - \alpha)\frac{\partial u}{\partial n}\) where \(\alpha \in [0, 1]\), \(g : \partial\Omega \rightarrow \mathbb{R}^+\) with \(g = 1\) when \(\alpha = 1\), and \(c, \lambda\) are positive constants, the weight \(m(x)\) satisfies \(m(x) \in C(\Omega), m(x) > 0\) for all \(x \in \Omega\) and \(h : \overline{\Omega} \rightarrow \mathbb{R}\) is a \(C^{1,\alpha}(\overline{\Omega})\) function satisfying \(h(x) > 0\) for \(x \in \Omega\), \(\max h(x) = 1\) for \(x \in \overline{\Omega}\) and \(h(x) = 0\) for \(x \in \partial\Omega\). We shall establish that every positive solution is linearly unstable.

Keywords: Diffusive logistic equation; harvesting; linearized stability of positive solutions; p-Laplacian.
Mathematics Subject Classification: 35J60, 35B30, 35B40

1 Introduction

In this paper, we consider the stability of positive solution to the boundary value problem

\[-\Delta_p u = \lambda m(x)(u^{\gamma-1} - u^{p-1} - ch(x)), \quad x \in \Omega,\]
\[Bu = 0, \quad x \in \partial\Omega,\]

where \(\Delta_p\) denotes the p-Laplacian operator defined by \(\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z);\) \(p > 1, \gamma(>p), \Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary
Equation (1) arises in the studies of population biology of one species. Here, \( u \) is the population density, \( \lambda m(x)(u^{\gamma-1} - u^{p-1}) \) represent the logistic growth and \( \lambda m(x)ch(x) \) representing the rate of harvesting (see [1, 4]). In [2], instability of such solutions was proven when \( p = 2 \) (the Laplacian operator), \( \gamma = 3 \) and \( c = 0 \) (non-harvesting case). The purpose of this paper is to extend this study to the \( p \)-Laplacian case with constant yield harvesting. However studying the instability of positive solutions in this case is significantly harder. For existence results of positive solutions for Eq. (1) see [3].

We recall that, if \( u \) be any nonnegative solution of

\[
\begin{align*}
-\Delta_p u &= g(x, u), \quad x \in \Omega, \\
0 &= \frac{\partial u}{\partial n}, \quad x \in \partial \Omega,
\end{align*}
\]

then the linearized equation about \( u \) is

\[
\begin{align*}
-(p-1) \text{div} (|\nabla u|^{p-2} \nabla \phi) - g_u(x, u) \phi &= \mu \phi, \quad x \in \Omega, \\
\phi &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( g_u(x, u) \) denotes the partial derivative of \( g(x, u) \) with respect to \( u \). Eq. (4) obtained from the formal derivative of the operator \( \Delta_p \).

**Definition 1.1.** We call a solution \( u \) of (3) a linearly stable solution if all eigenvalues of (4) are strictly positive, which can be inferred if the principal eigenvalue \( \mu_1 > 0 \). Otherwise \( u \) is linearly unstable.

## 2 Main result

In this section, we shall prove the instability of solution \( u \) by showing that the principal eigenvalue \( \mu_1 \), of the equation linearized about \( u \) is negative; the instability of \( u \) then follows from the well-known principle of linearized stability (see [5]). Our main result is formulate in the following theorem.

**Theorem 2.1.** Every positive solution of (1)-(2) is linearly unstable.

**Proof.** From (4) the linearized equation about \( u \) is

\[
-(p-1) \text{div} (|\nabla u|^{p-2} \nabla \phi) - \lambda m(x)[(\gamma-1)u^{\gamma-2} - (p-1)u^{p-2}]\phi = \mu \phi, \quad x \in \Omega,
\]
\[ B \phi = 0, \quad x \in \partial \Omega. \] (6)

Let \( \mu_1 \) be the principal eigenvalue and let \( \psi(x) (\geq 0) \) be a corresponding eigenfunction. Multiplying (1) by \((p - 1)\psi(x)\) and (5) by \(u\), then subtracting and integrating over \(\Omega\), we obtain

\[
(p - 1) \int_{\Omega} [u \text{div}(|\nabla u|^{p-2}\nabla \psi) - \psi(x) \text{div}(|\nabla u|^{p-2}\nabla u)] \, dx
\]

\[
+ \int_{\Omega} \lambda \psi(x) m(x) [(\gamma - p)u^{\gamma - 1} + (p - 1)ch(x)] \, dx
\]

\[
= - \mu_1 \int_{\Omega} \psi(x)u(x) \, dx. \] (7)

But by Green’s first identity

\[
\int_{\Omega} u \text{div}(|\nabla u|^{p-2}\nabla \psi) \, dx = \int_{\Omega} u|\nabla u|^{p-2}(\Delta \psi) \, dx + \int_{\Omega} u \nabla \psi \nabla(|\nabla u|^{p-2}) \, dx
\]

\[
= - \int_{\Omega} \nabla(u|\nabla u|^{p-2})\nabla \psi(x) \, dx
\]

\[
+ \int_{\Omega} u \nabla \psi \nabla(|\nabla u|^{p-2}) \, dx + \int_{\partial \Omega} u|\nabla u|^{p-2}(\frac{\partial \psi}{\partial n}) \, ds
\]

\[
= - \int_{\Omega} |\nabla u|^{p-2}(\nabla u \nabla \psi) \, dx + \int_{\partial \Omega} u|\nabla u|^{p-2}(\frac{\partial \psi}{\partial n}) \, ds, \] (8)

and

\[
\int_{\Omega} \psi(x) \text{div}(|\nabla u|^{p-2}\nabla u) \, dx = \int_{\Omega} \psi(x)|\nabla u|^{p-2}(\Delta u) \, dx + \int_{\Omega} \psi(x) \nabla u \nabla(|\nabla u|^{p-2}) \, dx
\]

\[
= - \int_{\Omega} \nabla(\psi|\nabla u|^{p-2}) \nabla u \, dx
\]

\[
+ \int_{\Omega} \psi(x) \nabla u \nabla(|\nabla u|^{p-2}) \, dx + \int_{\partial \Omega} \psi(x)|\nabla u|^{p-2}(\frac{\partial u}{\partial n}) \, ds
\]

\[
= - \int_{\Omega} |\nabla u|^{p-2}(\nabla u \nabla \psi) \, dx + \int_{\partial \Omega} \psi(x)|\nabla u|^{p-2}(\frac{\partial u}{\partial n}) \, ds. \] (9)

By using (8) – (9) in (7) we get

\[- \mu_1 \int_{\Omega} \psi(x)u(x) \, dx = \lambda \int_{\Omega} \psi(x) m(x) [(\gamma - p)u^{\gamma - 1} + (p - 1)ch(x)] \, dx\]
\[ + \int_{\partial \Omega} |\nabla u|^{p-2} [u (\frac{\partial \psi}{\partial n}) - \psi(s) (\frac{\partial u}{\partial n})] ds. \quad (10) \]

We notice that when \( \alpha = 1 \) (then \( h = 1 \)) we have \( Bu = u = 0 \) for \( s \in \partial \Omega \) and also we have \( \psi = 0 \) for \( s \in \partial \Omega \). Hence,

\[ \int_{\partial \Omega} |\nabla u|^{p-2} [u (\frac{\partial \psi}{\partial n}) - \psi(s) (\frac{\partial u}{\partial n})] ds = 0, \quad (11) \]

and when \( \alpha \neq 1 \), we have

\[ \int_{\partial \Omega} |\nabla u|^{p-2} [u (\frac{\partial \psi}{\partial n}) - \psi(s) (\frac{\partial u}{\partial n})] ds = \int_{\partial \Omega} |\nabla u|^{p-2} \left\{ \frac{\alpha g \psi(s)}{(1 - \alpha)} \right\} (u - u) ds = 0. \quad (12) \]

By using (11) - (12) in (10) we get

\[ -\mu_1 \int_{\Omega} \psi(x) u(x) dx = \lambda \int_{\Omega} \psi(x) m(x) [\gamma - p] u^{\gamma-1} + (p - 1) ch(x)] dx > 0. \quad (13) \]

But \( \psi > 0 \) for \( x \in \Omega \) and \( u > 0 \). Hence, it is easy to see that \( \mu_1 < 0 \) and the result follows (see [5]). \( \diamond \)

**References**


**Received: October 9, 2005**