Stability properties of positive solutions
for a boundary value problem

G. A. Afrouzi and S. H. Rasouli

Department of Mathematics, Faculty of Basic Sciences
Mazandaran University, Babolsar, Iran
afrouzi@umz.ac.ir

Abstract

We consider the boundary value problem

\[-\Delta u = \lambda m(x)f(u) + g(x,u), \quad x \in \Omega,\]
\[Bu = 0, \quad x \in \partial\Omega,\]

where \(\Delta\) denotes the Laplacian operator, \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary \(Bu(x) = \alpha h(x)u + (1 - \alpha)\frac{\partial u}{\partial n}\) where \(\alpha \in [0,1]\), \(h : \partial \Omega \rightarrow R^+\) with \(h = 1\) when \(\alpha = 1\), \(\lambda > 0\) is a constant, the weight \(m(x)\) satisfies \(m(x) \in C(\Omega), m(x) > 0\) for all \(x \in \Omega\), \(f \in C^2[0,\infty)\) and \(g : \Omega \times [0,\infty) \rightarrow R\) is a continuous function. We provide a simple proof to establish that every positive solution is linear unstable under certain conditions.

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1 Introduction

In this paper, we study the stability of positive solutions to the boundary value problem

\[-\Delta u = \lambda m(x)f(u) + g(x,u), \quad x \in \Omega,\]  \hspace{1cm} (1)
\[Bu = 0, \quad x \in \partial\Omega,\]  \hspace{1cm} (2)

where \(\Delta\) denotes the Laplacian operator, \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary \(Bu(x) = \alpha h(x)u + (1 - \alpha)\frac{\partial u}{\partial n}\) where \(\alpha \in [0,1]\), \(h : \partial \Omega \rightarrow R^+\) with \(h = 1\) when \(\alpha = 1\), i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, \(\lambda > 0\) is a constant, the weight
Theorem 1.1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a twice continuously differentiable function, then

(i) if \( f'' > 0 \) and \( f(0) \leq 0 \), then every nontrivial nonnegative solution of \( (3) \) is unstable. while

(ii) if \( f'' < 0 \) and \( f(0) \geq 0 \), then every nontrivial nonnegative solution of \( (3) \) is stable.

We call a function \( g \) strictly convex (or concave) if \( g'' \geq (g'' \leq) \), respectively, and not constant zero on any subinterval.

We recall that, if \( u \) be any nonnegative solution of

\[
\begin{cases}
-\Delta u = g(x, u), & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
\]

then the linearized equation about \( u \) is

\[
\begin{cases}
-\Delta \phi - g_u(x, u)\phi = \mu\phi, & x \in \Omega, \\
\phi = 0, & x \in \partial\Omega,
\end{cases}
\]

where \( g_u(x, u) \) denotes the partial derivative of \( g(x, u) \) with respect to \( u \).

Definition 1.2. We call a solution \( u \) of \( (4) \) a linearly stable solution if all eigenvalues of \( (5) \) are strictly positive, which can be inferred if the principal eigenvalue \( \mu_1 > 0 \). Otherwise \( u \) is linearly unstable.


\section{Stability results}

In this section, we shall prove the instability of positive solution \( u \) by showing that the principal eigenvalue \( \mu_1 \), of the equation linearized about \( u \) is negative, the instability of \( u \) then follows from the well-known principle of linearized stability (see [4]).

Our main result is formulate in the following theorem.

\textbf{Theorem 2.1.} If \( f'' > 0, f(0) \leq 0 \) and \( u \to g(x, u) \) be strictly convex and \( g(x, 0) \leq 0 \) for all fixed \( x \in \Omega \), then every positive solution of (1) – (2) is linearly unstable.

\textbf{Proof.} Let \( u \) be any nontrivial nonnegative stationary solution of (1) – (2), then from (5) the linearized equation about \( u \) is

\[
- \Delta \phi - [\lambda m(x)f'(u) - g_u(x, u)] \phi = \mu \phi, \quad x \in \Omega, \quad (6)
\]

\[
B \phi = 0, \quad x \in \partial \Omega. \quad (7)
\]

Let \( \mu_1 \) be the principal eigenvalue and let \( \psi(x) \geq 0 \) be a corresponding eigenfunction. Multiplying (1) by \( \psi(x) \) and (6) by \( u \), then subtracting and integrating over \( \Omega \), we obtain

\[
\int_{\Omega} [u\Delta \psi - \psi(x)\Delta u] \, dx + \lambda \int_{\Omega} m(x)\psi(x)[uf'(u) - f(u)] \, dx
\]

\[
+ \int_{\Omega} \psi(x)[ug_u(x, u) - g(x, u)] \, dx
\]

\[
= - \mu_1 \int_{\Omega} \psi(x)u(x) \, dx. \quad (8)
\]

But by green’s first identity

\[
\int_{\Omega} u\Delta \psi \, dx = - \int_{\Omega} (\nabla u \nabla \psi) \, dx + \int_{\partial \Omega} u(\frac{\partial \psi}{\partial n}) \, ds, \quad (9)
\]

and

\[
\int_{\Omega} \psi(x)\Delta u \, dx = - \int_{\Omega} (\nabla u \nabla \psi) \, dx + \int_{\partial \Omega} \psi(x)(\frac{\partial u}{\partial n}) \, ds. \quad (10)
\]

By using (9) – (10) in (8) we get

\[
- \mu_1 \int_{\Omega} \psi(x)u(x) \, dx = \lambda \int_{\Omega} m(x)\psi(x)[uf'(u) - f(u)] \, dx
\]

\[
+ \int_{\Omega} \psi(x)[ug_u(x, u) - g(x, u)] \, dx
\]
\[+ \int_{\partial \Omega} [u \left( \frac{\partial \psi}{\partial n} \right) - \psi(s) \left( \frac{\partial u}{\partial n} \right)] ds. \tag{11}\]

We notice that when \(\alpha = 1\) (then \(h = 1\)) we have \(Bu = u = 0\) for \(s \in \partial \Omega\) and also we have \(\psi = 0\) for \(s \in \partial \Omega\). Hence,

\[
\int_{\partial \Omega} [u \left( \frac{\partial \psi}{\partial n} \right) - \psi(s) \left( \frac{\partial u}{\partial n} \right)] ds = 0,
\tag{12}\]

and when \(\alpha \neq 1\), we have

\[
\int_{\partial \Omega} [u \left( \frac{\partial \psi}{\partial n} \right) - \psi(s) \left( \frac{\partial u}{\partial n} \right)] ds = \int_{\partial \Omega} \left\{ \frac{\alpha h \psi(s)}{(1 - \alpha)} \right\} (u - u) ds = 0. \tag{13}\]

By using (12) – (13) in (11) we get

\[-\mu_1 \int_{\Omega} \psi(x)u(x)dx = \lambda \int_{\Omega} m(x)\psi(x)[uf'(u) - f(u)]dx
\]

\[+ \int_{\Omega} \psi(x)[ug_u(x, u) - g(x, u)]dx \tag{14}\]

Our assumption implies that \(uf'(u) - f(u) > 0\) for \(u \in \mathbb{R}^+\) also, \(ug_u(x, u) - g(x, u) > 0\) for \(u \in \mathbb{R}^+\). Thus, we have

\[-\mu_1 \int_{\Omega} \psi(x)u(x)dx > 0. \tag{15}\]

Hence, it is easy to see that \(\mu_1 < 0\) and the result follows (see [4]).

\[\Diamond\]

**Corollary 2.3.** Assume that in Theorem 2.1 we have \(f'' < 0\), \(f(0) \geq 0\) and \(u \rightarrow g(x, u)\) be strictly concave and \(g(x, 0) \geq 0\) for all fixed \(x \in \Omega\), then every positive solution of (1) – (2) is linearly stable.

**Proof.** The proof proceeding is identical to the proof of Theorem 2.1. In fact, instead of (15) we get

\[-\mu_1 \int_{\Omega} \psi(x)u(x)dx < 0, \tag{16}\]

but \(\psi > 0\) for \(x \in \Omega\), \(u > 0\) and hence \(\mu_1 > 0\). This completes the proof.

\[\Diamond\]

**Remark 1.2.** Recently in [6], the author study the stability of nonnegative stationary solutions of symmetric cooperative semilinear systems with some convex (resp, concave) nonlinearity condition.
References


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