Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 7, 333 - 340

On Hardy Type Integral Inequality Associated with The Generalized Translation

Mehmet Zeki SARIKAYA, Hüseyin YILDIRIM and Aziz SAGLAM

Department of Mathematics, Faculty of Science and Arts
Kocatepe University, Afyon, Turkey
sarikaya@aku.edu.tr, hyildir@aku.edu.tr azizsaglam@aku.edu.tr

Abstract. In this study, we obtain a new proofs of some Hardy-type inequalities associated with generalized translation.

Keywords: Shift Operators, Hardy integral inequalities and Hölder inequality
Mathematics Subject Classification: 26D15

1. Introduction

In [1], Hardy proved the following inequality. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $u \geq 0$ and $0 < \int\limits_0^\infty u^p(t)dt < \infty$, then

$$\int\limits_0^\infty \left(\frac{1}{x} \int\limits_0^x u(t)dt \right)^p dx \quad < \quad q^p \int\limits_0^\infty u^p(t)dt \tag{1.1}$$

where the constant $q = p\,(p-1)^{-1}$ is the best possible. This inequality plays an important role in analysis and its applications. It is obvious that, for parameters a and b such that $0 < a < b < \infty$, the following inequality is also valid

$$\int\limits_a^b \left(\frac{1}{x} \int\limits_0^x u(t)dt \right)^p dx \quad < \quad q^p \int\limits_a^b u^p(t)dt \tag{1.2}$$

where $0 < \int\limits_0^\infty u^p(t)dt < \infty$. The classical Hardy inequality asserts that if $p > 1$ and u is a non-negative measurable function on (a, b), then (1.2) is true unless $u = 0$ a.e. in (a, b)[5,6]. This inequality remains true provided that $0 < a < b < \infty$.

In particular, Hardy[2] in 1928 gave a generalized form of inequality (1.1) when he showed that for any $r \neq 1, p > 1$ and any integrable function $u(x) \geq 0$

on $(0,\infty)$ for which

$$F(x) = \begin{cases} \int\limits_0^x u(t)dt & \text{for} \quad r>1 \\ \int\limits_x^\infty u(t)dt & \text{for} \quad r<1 \end{cases}$$

then

$$\int\limits_0^\infty x^{-m}F^p(x)dx < \left(\tfrac{p}{|r-1|}\right)^p \int\limits_0^\infty x^{-m}\left[xu(x)\right]^p dx \tag{1.3}$$

unless $u=0$, where the constant is also best possible.

The aim of the paper is to establish some Hardy-type inequalities similarly (1.3) associated with Shift operator. These inequalities generalize some known results and simplify the proofs of some existing results. Troughout this paper, functions are assumed to be measurable, locally integrable and the left-hand sides of the inequalities if the right-hand sides exist.

We give some notations and definitions: $L_{p,v} = L_{p,v}(\mathbf{R}^+)$ is defined with respect to the Lebesgue measure $x^{2v}dx$ the following

$$L_{p,v} = L_{p,v}(R^+) = \left\{ f : \|f\|_{p,v} = \left(\int\limits_{\mathbf{R}^+} |f(x)|^p x^{2v} dx \right)^{\frac{1}{p}} < \infty \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

where $1 \leq p < \infty$ and $v > 0$.

Denote by T^y the generalized shift operator acting according to the law

$$T^y f(t) = C_v \int\limits_0^\pi f\left(\sqrt{y^2+t^2-2ty\cos\theta}\right)\sin^{2v-1}\theta d\theta \tag{1.4}$$

where $t, y \in \mathbf{R}^+$, $C_v = \dfrac{\Gamma(v+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(v)}$ [4] and [7]. We remark that this shift operator is closely connected with the Bessel differential operator [3].

$$\begin{aligned} \frac{d^2u}{dt^2} + \frac{2v}{t}\frac{du}{dt} &= \frac{d^2u}{dy^2} + \frac{2v}{y}\frac{du}{dy} \\ u(t,0) &= f(t) \\ u_y(t,0) &= 0 \end{aligned}$$

In this paper, we consider the following equality for $y=1$

$$Tf(t) = C_v \int\limits_0^\pi f\left(\sqrt{1+t^2-2t\cos\theta}\right)\sin^{2v-1}\theta d\theta.$$

Remark 1.1. Observe that the case $\cos\theta = \frac{1}{2t}$ we have

$$Tf(t) = C_v f(t) \int\limits_0^\pi \sin^{2v-1}\theta d\theta = f(t).$$

2. Main Results

Theorem 2.1. Let $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ v > 0,\ r > 2v + 1$ and $u(x) \geq 0$ so that

$$1 + \frac{p(2v+1)}{r - 2v - 1} \geq \frac{1}{\lambda} > 0$$

almost everywhere for some $\lambda > 0$. For any $a \in (0, \infty)$, if

$$F(x) := \frac{1}{x^{2v+1}} \int_a^x Tu(t) t^{2v} dt, \quad x \in (0, \infty)$$

then for $b \geq a$

$$\int_a^b x^{-r} F^p(x)\, x^{2v} dx \leq \left(\tfrac{\lambda p}{r-2v-1}\right)^p \int_a^b x^{-r} \left[Tu(x)\right]^p x^{2v} dx \tag{2.1}$$

Proof. Integrating the left-hand side of inequality (2.1) by parts gives

$$\begin{aligned} \int_a^b x^{-r} F^p(x)\, x^{2v} dx &= \tfrac{b^{-r+2v+1}}{-r+2v+1} F^p(b) + \int_a^b \tfrac{p(2v+1)}{-r+2v+1} x^{-r} F^p(x)\, x^{2v} dx \\ &- \int_a^b \tfrac{p}{-r+2v+1} x^{-r} F^{p-1}(x)\, Tu(x)\, x^{2v} dx. \end{aligned}$$

Using the fact that $r > 2v + 1$ and $F(b) \geq 0$ we have

$$\int_a^b x^{-r} F^p(x) \left[1 + \frac{p(2v+1)}{r-2v-1}\right] x^{2v} dx$$

$$= \frac{b^{-r+2v+1}}{-r+2v+1} F^p(b) + \frac{p}{r-2v-1} \int_a^b x^{-r} F^{p-1}(x)\, Tu(x)\, x^{2v} dx$$

$$\leq \frac{p}{r-2v-1} \int_a^b x^{-r} F^{p-1}(x)\, Tu(x)\, x^{2v} dx.$$

Here, using the assumptation on λ, we have

$$\begin{aligned} \int_a^b x^{-r} F^p(x) \tfrac{1}{\lambda} x^{2v} dx &\leq \int_a^b x^{-r} F^p(x) \left[1 + \tfrac{p(2v+1)}{r-2v-1}\right] x^{2v} dx \\ &\leq \tfrac{p}{r-2v-1} \int_a^b \left[x^{(-r+2v)\frac{1}{p}} Tu(x)\right] \left[x^{(-r+2v)\frac{1}{q}} F^{p-1}(x)\right] dx. \end{aligned}$$

By Hölder's inequality

$$\int_a^b x^{-r} F^p(x)\, x^{2v} dx \leq \tfrac{\lambda p}{r-2v-1} \left[\int_a^b x^{-r} \left[Tu(x)\right]^p x^{2v} dx\right]^{\frac{1}{p}} \left[\int_a^b x^{-r} F^p(x)\, x^{2v} dx\right]^{\frac{1}{q}}.$$

From this, it follows that

$$\int_a^b x^{-r} F^p(x)\, x^{2v} dx \leq \left(\tfrac{\lambda p}{r-2v-1}\right)^p \int_a^b x^{-r} \left[Tu(x)\right]^p x^{2v} dx.$$

This proves the theorem.

Theorem 2.2. Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \upsilon>0,\ r>2\upsilon+1$ and $u(x)\geq 0$ so that

$$1+\frac{p(2\upsilon+1)}{r-2\upsilon-1}\geq\frac{1}{\lambda}>0$$

almost everywhere for some $\lambda>0$. If

$$F(x):=\frac{1}{x^{2\upsilon+1}}\int_{\frac{x}{2}}^{x}Tu(t)t^{2\upsilon}dt,\ \ x\in(0,\infty)$$

then for any $a\in(0,\infty)$ and $b\geq a$

(2.2)

$$\int_a^b x^{-r}F^p(x)\,x^{2\upsilon}dx \leq \left(\tfrac{\lambda p}{r-2\upsilon-1}\right)^p\int_a^b x^{-r}\left|Tu(x)-\tfrac{1}{2^{2\upsilon+1}}Tu\left(\tfrac{x}{2}\right)\right|^p x^{2\upsilon}dx$$

Proof. Integrating the left-hand side of inequality (2.2) by parts we obtain

$$\begin{aligned}\int_a^b x^{-r}F^p(x)\,x^{2\upsilon}dx &= \tfrac{b^{-r+2\upsilon+1}}{-r+2\upsilon+1}F^p(b)+\tfrac{p(2\upsilon+1)}{r-2\upsilon-1}\int_a^b x^{-r}F^p(x)\,x^{2\upsilon}dx\\ - &- \int_a^b\tfrac{p}{-r+2\upsilon+1}x^{-r}\left[Tu(x)-\tfrac{1}{2^{2\upsilon+1}}Tu\left(\tfrac{x}{2}\right)\right]F^{p-1}(x)\,x^{2\upsilon}dx.\end{aligned}$$

Using the fact that $r>2\upsilon+1$ and $F(b)\geq 0$ we have

$$\int_a^b x^{-r}F^p(x)\left[1+\frac{p(2\upsilon+1)}{r-2\upsilon-1}\right]x^{2\upsilon}dx$$

$$=\frac{b^{-r+2\upsilon+1}}{-r+2\upsilon+1}F^p(b)$$

$$+\frac{p}{r-2\upsilon-1}\int_a^b x^{-r}\left[Tu(x)-\frac{1}{2^{2\upsilon+1}}Tu\left(\frac{x}{2}\right)\right]F^{p-1}(x)\,x^{2\upsilon}dx$$

$$\leq\frac{p}{r-2\upsilon-1}\int_a^b x^{-r}\left[Tu(x)-\frac{1}{2^{2\upsilon+1}}Tu\left(\frac{x}{2}\right)\right]F^{p-1}(x)\,x^{2\upsilon}dx.$$

Here, using the assumptation on λ, we have

$$\int_a^b x^{-r}F^p(x)\frac{1}{\lambda}x^{2\upsilon}dx\leq\int_a^b x^{-r}F^p(x)\left[1+\frac{p(2\upsilon+1)}{r-2\upsilon-1}\right]x^{2\upsilon}dx$$

$$\leq\frac{p}{r-2\upsilon-1}\int_a^b\left[\left(x^{-r+2\upsilon}\right)^{\frac{1}{p}}\left[Tu(x)-\frac{1}{2^{2\upsilon+1}}Tu\left(\frac{x}{2}\right)\right]\right]\left[\left(x^{-r+2\upsilon}\right)^{\frac{1}{q}}F^{p-1}(x)\right]dx.$$

By Hölder's inequality we have

$$\int_a^b x^{-r}F^p(x)\,x^{2\upsilon}dx$$

$$\leq \frac{\lambda p}{r-2v-1}\left[\int_a^b x^{-r}\left|Tu\left(x\right)-\frac{1}{2^{2v+1}}Tu\left(\frac{x}{2}\right)\right|^p x^{2v}dx\right]^{\frac{1}{p}}\left[\int_a^b x^{-r}F^p\left(x\right)x^{2v}dx\right]^{\frac{1}{q}}.$$

From this, it follows that

$$\int_a^b x^{-r}F^p\left(x\right)x^{2v}dx \quad\leq\quad \left(\tfrac{\lambda p}{r-2v-1}\right)^p \int_a^b x^{-r}\left|Tu\left(x\right)-\tfrac{1}{2^{2v+1}}Tu\left(\tfrac{x}{2}\right)\right|^p x^{2v}dx.$$

The proof is completed.

Theorem 2.3. Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ v>0,\ r>2v+1$ and $u(x)\geq 0$ so that

$$1+\frac{p(2v+1)}{r-2v-1}+\frac{p}{r-2v-1}x\frac{[Tu(x)]'}{[Tu(x)]}\geq\frac{1}{\lambda}>0$$

almost everywhere for some $\lambda>0$. For any $a\in(0,\infty)$, if

$$F(x):=\frac{1}{x^{2v+1}[Tu(x)]}\int_{\frac{x}{2}}^{x}Tu(t)Tv(t)t^{2v}dt,\quad x\in(0,\infty)$$

then for $b\geq a$ we have

$$\int_a^b x^{-r}F^p\left(x\right)x^{2v}dx \quad\leq\quad \left(\tfrac{\lambda p}{r-2v-1}\right)^p\int_a^b x^{-r}\left[Tv\left(x\right)\right]^p x^{2v}dx. \tag{2.3}$$

Proof. Integrating the left-hand side of inequality (2.3) by parts, we have

$$\begin{aligned}
\int_a^b x^{-r}F^p\left(x\right)x^{2v}dx &= \tfrac{b^{-r+2v+1}}{-r+2v+1}F^p\left(b\right)-\tfrac{p(2v+1)}{r-2v-1}\int_a^b x^{-r}F^p\left(x\right)x^{2v}dx\\
&+ \tfrac{p}{r-2v-1}\int_a^b x^{-r+1}\tfrac{[Tu(x)]'}{Tu(x)}F^p\left(x\right)x^{2v}dx\\
&+ \tfrac{p}{r-2v-1}\int_a^b x^{-r}\left[Tv\left(x\right)\right]F^{p-1}\left(x\right)x^{2v}dx.
\end{aligned}$$

Using the fact that $r>2v+1,\ F(b)\geq 0$ and assumption on λ, we have

$$\begin{aligned}
\int_a^b x^{-r}F^p\left(x\right)\tfrac{1}{\lambda}x^{2v}dx &\leq \int_a^b x^{-r}F^p\left(x\right)\left[1+\tfrac{p(2v+1)}{r-2v-1}-\tfrac{p}{r-2v-1}x\tfrac{[Tu(x)]'}{Tu(x)}\right]x^{2v}dx\\
&= \tfrac{b^{-r+2v+1}}{-r+2v+1}F^p\left(b\right)+\tfrac{p}{r-2v-1}\int_a^b x^{-r}F^{p-1}\left(x\right)\left[Tv\left(x\right)\right]x^{2v}dx\\
&\leq \tfrac{p}{r-2v-1}\int_a^b x^{-r}F^{p-1}\left(x\right)\left[Tv\left(x\right)\right]x^{2v}dx.
\end{aligned}$$

By Hölder's inequality

$$\int_a^b x^{-r}F^p\left(x\right)x^{2v}dx$$

$$\leq \frac{\lambda p}{r-2\upsilon-1}\left[\int_a^b x^{-r}\left[T\upsilon(x)\right]^p x^{2\upsilon}dx\right]^{\frac{1}{p}}\left[\int_a^b x^{-r}F^{(p-1)q}(x)\,x^{2\upsilon}dx\right]^{\frac{1}{q}}.$$

From this, it follows that

$$\int_a^b x^{-r}F^p(x)\,x^{2\upsilon}dx \quad\leq\quad \left(\frac{\lambda p}{r-2\upsilon-1}\right)^p \int_a^b x^{-r}\left[T\upsilon(x)\right]^p x^{2\upsilon}dx.$$

This proves the theorem.

Theorem 2.4. Let $p > q > 0$, $\alpha \geq 0$ $\frac{1}{p}+\frac{1}{q}=1$, $\upsilon > 0$ and $r > 2\upsilon+1$. Let $Tu : (0,\infty) \to (0,\infty)$ be absolutely continuous and let $T\upsilon : [0,\infty) \to [0,\infty)$ be integrable so that

$$1+\frac{p}{q}\frac{2\upsilon}{r-2\upsilon-1}+\frac{p}{q}\frac{1}{r-2\upsilon-1}\alpha x\frac{[Tu(x)]'}{[Tu(x)]} \geq \frac{1}{\lambda} > 0$$

almost everywhere for some $\lambda > 0$. For any $a \in (0,\infty)$, if

$$F(x) := \frac{1}{x^{2\upsilon}[Tu(x)]^\alpha}\int_a^x \frac{Tu(t)T\upsilon(t)}{t}t^{2\upsilon}dt, \quad x \in (0,\infty)$$

then for $b \geq a$, we have

$$\int_a^b x^{-r}F^{\frac{p}{q}}(x)\,x^{2\upsilon}dx \quad\leq\quad \left(\frac{p}{q}\frac{\lambda}{-r+2\upsilon+1}\right)^{\frac{p}{q}} \int_a^b x^{-r}\frac{[T\upsilon(x)]^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}}x^{2\upsilon}dx. \tag{2.4}$$

Proof. Integrating the left-hand side of inequality (2.4) by parts, we have

$$\begin{aligned}\int_a^b x^{-r}F^p(x)\,x^{2\upsilon}dx \quad&=\quad \frac{b^{-r+2\upsilon+1}}{-r+2\upsilon+1}F^{\frac{p}{q}}(b)+\int_a^b \frac{p}{q}\frac{2\upsilon}{-r+2\upsilon+1}x^{-r}F^{\frac{p}{q}}(x)\,x^{2\upsilon}dx\\ &+\quad \int_a^b \alpha\frac{p}{q}\frac{x^{-r+1}}{-r+2\upsilon+1}F^{\frac{p}{q}}(x)\,\frac{[Tu(x)]'}{Tu(x)}x^{2\upsilon}dx\\ &-\quad \int_a^b \frac{p}{q}\frac{x^{-r}}{-r+2\upsilon+1}\frac{T\upsilon(x)}{[Tu(x)]^{\alpha-1}}F^{\frac{p}{q}-1}(x)\,x^{2\upsilon}dx\end{aligned}$$

Using the fact that $r > 2\upsilon+1$, $F(b) \geq 0$ and assumption on λ, we have

$$\int_a^b x^{-r}F^{\frac{p}{q}}(x)\,\frac{1}{\lambda}x^{2\upsilon}dx$$

$$\leq \int_a^b x^{-r}F^{\frac{p}{q}}(x)\left[1+\frac{p}{q}\frac{2\upsilon}{r-2\upsilon-1}+\alpha\frac{p}{q}\frac{1}{r-2\upsilon-1}x\frac{[Tu(x)]'}{Tu(x)}\right]x^{2\upsilon}dx$$

$$= \frac{b^{-r+2\upsilon+1}}{-r+2\upsilon+1}F^{\frac{p}{q}}(b)+\frac{p}{q}\frac{1}{-r+2\upsilon+1}\int_a^b x^{-r}\frac{T\upsilon(x)}{[Tu(x)]^{\alpha-1}}F^{\frac{p}{q}-1}(x)\,x^{2\upsilon}dx$$

$$\leq \frac{p}{q}\frac{1}{-r+2\upsilon+1}\int_a^b x^{-r}\frac{T\upsilon(x)}{[Tu(x)]^{\alpha-1}}F^{\frac{p}{q}-1}(x)\,x^{2\upsilon}dx.$$

Using the application of Hölder's inequality with indices $\frac{p}{q}$ and $\frac{p}{p-q}$ respectively, we obtain

$$\int_a^b x^{-r} F^{\frac{p}{q}}(x)\, x^{2v} dx$$

$$\leq \frac{p}{q} \frac{\lambda}{-r+2v+1} \left[\int_a^b x^{-r} \frac{[Tv(x)]^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}} x^{2v} dx \right]^{\frac{q}{p}} \left[\int_a^b x^{-r} F^{\frac{p}{q}}(x)\, x^{2v} dx \right]^{\frac{p-q}{p}}.$$

From this, it follows that

$$\int_a^b x^{-r} F^{\frac{p}{q}}(x)\, x^{2v} dx \leq \left(\frac{p}{q} \frac{\lambda}{-r+2v+1} \right)^{\frac{p}{q}} \int_a^b x^{-r} \frac{[Tv(x)]^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}} x^{2v} dx.$$

This proves the theorem.

Theorem 2.5. Let $p > q > 0$, $\alpha \geq 0$ $\frac{1}{p} + \frac{1}{q} = 1$, $v > 0$ and $r < 2v+1$. Let $Tu : (0, \infty) \to (0, \infty)$ be absolutely continuous and let $Tv : [0, \infty) \to [0, \infty)$ be integrable so that

$$1 - \frac{p}{q} \frac{2v}{r - 2v - 1} - \frac{p}{q} \frac{1}{r - 2v - 1} \alpha x \frac{[Tu(x)]'}{[Tu(x)]} \geq \frac{1}{\lambda} > 0$$

a.e for some $\lambda > 0$. For any $a \in (0, \infty)$, if

$$F(x) := \frac{1}{x^{2v} [Tu(x)]^{\alpha}} \int_a^x \frac{Tu(t) Tv(t)}{t} t^{2v} dt, \quad x \in (0, \infty)$$

then for $b \geq a$, we have

$$\int_a^b x^{-r} F^{\frac{p}{q}}(x)\, x^{2v} dx \leq \left(\frac{p}{q} \frac{\lambda}{-r+2v+1} \right)^{\frac{p}{q}} \int_a^b x^{-r} \frac{[Tv(x)]^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}} x^{2v} dx. \tag{2.5}$$

Proof. This is similar to the proof of Theorem 2.4.

Theorem 2.6. Let $p > q > 0$, $\alpha \geq 0$ $\frac{1}{p} + \frac{1}{q} = 1$, $v > 0$ and $r > 2v+1$. Let $Tu : (0, \infty) \to (0, \infty)$ be absolutely continuous and let $Tv : [0, \infty) \to [0, \infty)$ be integrable so that

$$1 + \frac{p}{q} \frac{2v}{r - 2v - 1} + \frac{p}{q} \frac{1}{r - 2v - 1} \alpha x \frac{[Tu(x)]'}{[Tu(x)]} \geq \frac{1}{\lambda} > 0$$

a.e for some $\lambda > 0$. For any $a \in (0, \infty)$, if

$$F(x) := \frac{1}{x^{2v+1} [Tu(x)]^{\alpha}} \int_a^x Tu(t) Tv(t) t^{2v} dt, \quad x \in (0, \infty)$$

then for $b \geq a$, we have

$$\int_a^b x^{-r} F^{p}(x)\, x^{2v} dx \leq \left(\frac{p}{q} \frac{\lambda}{-r+2v+1} \right)^{p} \int_a^b x^{-r} \frac{[Tv(x)]^{p}}{[Tu(x)]^{(\alpha-1)p}} x^{2v} dx. \tag{2.6}$$

Proof. This is similar to the proof of Theorem 2.3 and Theorem 2.4.

References

[1] Hardy, G.H.Littlewood, J.E and Polya, G.: Inequalities, Cambridge Univ. Press, Cambridge, 1952.
[2] Hardy, G. H.: Notes on some points in the integral calculus, Messenger Math. 57(1928), 12-16.
[3] Levitan, B.M.: Expansion in Fourier Series and Integrals with Bessel Functions, PUspeki, Mat., Nauka (N.S) 6, No:2(42), pp.102-143 (in Russian) (1952).
[4] Levitan, B.M.: Generelized Translation Operators and Some of Their Applications, Moscova 1962 (Translation 1964).
[5] Pachpatte B.G.: On Some Generalizations of Hardy's Integral Inequality, Jour. Math. Anal. Appl. 234, no.1, 15-30, 1999.
[6] Yang, B., Zeng, Z. and Debnath, L.: On New Generalizations of Hardy's Integral Inequality, Jour. Math. Anal. Appl. 217, no.1, 321-327, 1998.
[7] Yıldırım, H.: *Riesz Potentials Generated by a Generalized shift operator.*Ankara Uni. Graduate school of Natural and Applieds Sciences Department of Math.Ph.D.thesis 1995.

Received: January 19, 2005