On Hardy Type Integral Inequality Associated with The Generalized Translation

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Abstract. In this study, we obtain a new proofs of some Hardy-type inequalities associated with generalized translation.

Keywords: Shift Operators, Hardy integral inequalities and Hölder inequality
Mathematics Subject Classification: 26D15

1. Introduction

In [1], Hardy proved the following inequality. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $u \geq 0$ and $0 < \int_0^\infty u^p(t)dt < \infty$, then

\begin{equation}
\int_0^\infty \left( \frac{1}{x} \int_0^x u(t)dt \right)^p dx < q^p \int_0^\infty u^p(t)dt
\end{equation}

where the constant $q = p(p - 1)^{-1}$ is the best possible. This inequality plays an important role in analysis and its applications. It is obvious that, for parameters $a$ and $b$ such that $0 < a < b < \infty$, the following inequality is also valid

\begin{equation}
\int_a^b \left( \frac{1}{x} \int_0^x u(t)dt \right)^p dx < q^p \int_a^b u^p(t)dt
\end{equation}

where $0 < \int_0^\infty u^p(t)dt < \infty$. The classical Hardy inequality asserts that if $p > 1$ and $u$ is a non-negative measurable function on $(a, b)$, then (1.2) is true unless $u = 0$ a.e. in $(a, b)[5,6]$. This inequality remains true provided that $0 < a < b < \infty$.

In particular, Hardy[2] in 1928 gave a generalized form of inequality (1.1) when he showed that for any $r \neq 1$, $p > 1$ and any integrable function $u(x) \geq 0$
on \((0, \infty)\) for which
\[
F(x) = \begin{cases} 
\int_0^x u(t) dt & \text{for } r > 1 \\
\int_x^\infty u(t) dt & \text{for } r < 1
\end{cases}
\]
then
\[
\int_0^\infty x^{-m} F^p(x) dx < \left( \frac{p}{p-1} \right)^p \int_0^\infty x^{-m} [xu(x)]^p dx
\]
unless \(u = 0\), where the constant is also best possible.

The aim of the paper is to establish some Hardy-type inequalities similarly (1.3) associated with Shift operator. These inequalities generalize some known results and simplify the proofs of some existing results. Throughout this paper, functions are assumed to be measurable, locally integrable and the left-hand sides of the inequalities if the right-hand sides exist.

We give some notations and definitions: \(L_{p,v} = L_{p,v}(\mathbb{R}^+), v > 0\) is defined with respect to the Lebesgue measure \(x^2 v dx\) the following
\[
L_{p,v} = \{ f : ||f||_{p,v} = \left( \int_{\mathbb{R}^+} |f(x)|^p x^{2v} dx \right)^{\frac{1}{p}} < \infty \}, \quad \frac{1}{p} + \frac{1}{q} = 1
\]
where \(1 \leq p < \infty\) and \(v > 0\).

Denote by \(T^y\) the generalized shift operator acting according to the law
\[
T^y f(t) = C_v \int_0^\pi f(\sqrt{y^2 + t^2 - 2ty \cos \theta}) \sin^{2v-1} \theta d\theta
\]
where \(t, y \in \mathbb{R}^+, C_v = \frac{\Gamma(v + \frac{1}{2})}{\Gamma\left(\frac{1}{2}\right) \Gamma(v)}\) [4] and [7]. We remark that this shift operator is closely connected with the Bessel differential operator [3].

\[
\frac{d^2 u}{dt^2} + \frac{2v}{t} \frac{du}{dt} = \frac{d^2 u}{dy^2} + \frac{2v}{y} \frac{du}{dy} \\
u(t, 0) = f(t) \\
u_y(t, 0) = 0
\]
In this paper, we consider the following equality for \(y = 1\)
\[
T f(t) = C_v \int_0^\pi f(\sqrt{1 + t^2 - 2t \cos \theta}) \sin^{2v-1} \theta d\theta.
\]

**Remark 1.1.** Observe that the case \(\cos \theta = \frac{1}{2t}\) we have
\[
T f(t) = C_v f(t) \int_0^\pi \sin^{2v-1} \theta d\theta = f(t).
\]
2. Main Results

Theorem 2.1. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $v > 0$, $r > 2v + 1$ and $u(x) \geq 0$ so that

$$1 + \frac{p(2v + 1)}{r - 2v - 1} \geq \frac{1}{\lambda} > 0$$

almost everywhere for some $\lambda > 0$. For any $a \in (0, \infty)$, if

$$F(x) := \frac{1}{x^{2v+1}} \int_a^x Tu(t)t^{2v}dt, \ x \in (0, \infty)$$

then for $b \geq a$

$$J_a^b x^{-r} F^p (x) x^{2v}dx \leq \left( \frac{\lambda p}{r - 2v - 1} \right)^p \int_a^b x^{-r} [Tu(x)]^p x^{2v}dx \quad (2.1)$$

Proof. Integrating the left-hand side of inequality (2.1) by parts gives

$$J_a^b x^{-r} F^p (x) x^{2v}dx = \frac{b^{-r+2v+1}}{-r+2v+1} F^p (b) + \int_a^b \frac{p(2v+1)}{-r+2v+1} x^{-r} F^p (x) x^{2v}dx$$

$$- \frac{p}{r - 2v - 1} \int_a^b x^{-r} F^{p-1} (x) Tu(x) x^{2v}dx.$$

Using the fact that $r > 2v + 1$ and $F(b) \geq 0$ we have

$$\int_a^b x^{-r} F^p (x) \left[ 1 + \frac{p(2v + 1)}{r - 2v - 1} \right] x^{2v}dx$$

$$= \frac{b^{-r+2v+1}}{-r+2v+1} F^p (b) + \int_a^b x^{-r} F^{p-1} (x) Tu(x) x^{2v}dx$$

$$\leq \frac{p}{r - 2v - 1} \int_a^b x^{-r} F^{p-1} (x) Tu(x) x^{2v}dx.$$

Here, using the assumption on $\lambda$, we have

$$\int_a^b x^{-r} F^p (x) \frac{1}{\lambda} x^{2v}dx \leq \int_a^b x^{-r} F^p (x) \left[ 1 + \frac{p(2v + 1)}{r - 2v - 1} \right] x^{2v}dx$$

$$\leq \frac{p}{r - 2v - 1} \int_a^b \left[ x^{(-r+2v)\frac{1}{p}} Tu(x) \right] \left[ x^{(-r+2v)\frac{2}{q}} F^{p-1}(x) \right] dx.$$

By Hölder’s inequality

$$\int_a^b x^{-r} F^p (x) x^{2v}dx \leq \frac{\lambda p}{r - 2v - 1} \left[ \int_a^b x^{-r} [Tu(x)]^p x^{2v}dx \right]^\frac{1}{p} \left[ \int_a^b x^{-r} F^p (x) x^{2v}dx \right]^\frac{1}{q}.$$

From this, it follows that

$$\int_a^b x^{-r} F^p (x) x^{2v}dx \leq \left( \frac{\lambda p}{r - 2v - 1} \right)^p \int_a^b x^{-r} [Tu(x)]^p x^{2v}dx.$$

This proves the theorem.
Theorem 2.2. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $v > 0$, $r > 2v + 1$ and $u(x) \geq 0$ so that

$$1 + \frac{p(2v + 1)}{r - 2v - 1} \geq \frac{1}{\lambda} > 0$$

almost everywhere for some $\lambda > 0$. If

$$F(x) := \frac{1}{x^{2v+1}} \int_{\frac{x}{2}}^{x} Tu(t)t^{2v}dt, \ x \in (0, \infty)$$

then for any $a \in (0, \infty)$ and $b \geq a$

(2.2) $$\int_{a}^{b} x^{-r} F^p(x) x^{2v} dx \leq \left(\frac{\lambda p}{r-2v-1}\right)^p \int_{a}^{b} x^{-r} |Tu(x) - \frac{1}{2^{2v+1}} Tu\left(\frac{x}{2}\right)|^p x^{2v} dx$$

Proof. Integrating the left-hand side of inequality (2.2) by parts we obtain

$$\int_{a}^{b} x^{-r} F^p(x) x^{2v} dx = \frac{b^{-r+2v+1}}{-r+2v+1} F^p(b) + \frac{p(2v+1)}{r-2v-1} \int_{a}^{b} x^{-r} F^p(x) x^{2v} dx$$

$$- \int_{a}^{b} \frac{p}{r-2v-1} x^{-r} \left[Tu(x) - \frac{1}{2^{2v+1}} Tu\left(\frac{x}{2}\right)\right] F^{p-1}(x) x^{2v} dx.$$

Using the fact that $r > 2v + 1$ and $F(b) \geq 0$ we have

$$\int_{a}^{b} x^{-r} F^p(x) \left[1 + \frac{p(2v + 1)}{r - 2v - 1}\right] x^{2v} dx$$

$$= \frac{b^{-r+2v+1}}{-r+2v+1} F^p(b)$$

$$+ \frac{p}{r - 2v - 1} \int_{a}^{b} x^{-r} \left[Tu(x) - \frac{1}{2^{2v+1}} Tu\left(\frac{x}{2}\right)\right] F^{p-1}(x) x^{2v} dx$$

$$\leq \frac{p}{r - 2v - 1} \int_{a}^{b} x^{-r} \left[Tu(x) - \frac{1}{2^{2v+1}} Tu\left(\frac{x}{2}\right)\right] F^{p-1}(x) x^{2v} dx.$$

Here, using the assumption on $\lambda$, we have

$$\int_{a}^{b} x^{-r} F^p(x) \frac{1}{\lambda} x^{2v} dx \leq \int_{a}^{b} x^{-r} F^p(x) \left[1 + \frac{p(2v + 1)}{r - 2v - 1}\right] x^{2v} dx$$

$$\leq \frac{p}{r - 2v - 1} \int_{a}^{b} \left[(x^{-r+2v})^\frac{1}{p} F^{p-1}(x)\right] \left[(x^{-r+2v})^\frac{1}{p} F^{p-1}(x)\right] dx.$$

By Hölder’s inequality we have

$$\int_{a}^{b} x^{-r} F^p(x) x^{2v} dx$$
The proof is completed.

**Theorem 2.3.** Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $v > 0$, $r > 2v + 1$ and $u(x) \geq 0$ so that

$$1 + \frac{p(2v + 1)}{r - 2v - 1} + \frac{p}{r - 2v - 1} x \frac{[Tu(x)]'}{[Tu(x)]} \geq \frac{1}{\lambda} > 0$$

almost everywhere for some $\lambda > 0$. For any $a \in (0, \infty)$, if

$$F(x) := \frac{1}{x^{2v+1}[Tu(x)]} \int_{\frac{x}{2}}^{x} Tu(t)Tv(t)t^{2v}dt, \ x \in (0, \infty)$$

then for $b \geq a$ we have

$$(2.3) \quad \int_{a}^{b} x^{-r}Fp(x)x^{2v}dx \leq \left(\frac{\lambda p}{r - 2v - 1}\right)^{p} \int_{a}^{b} x^{-r} [Tv(x)]^{p} x^{2v}dx.$$  

**Proof.** Integrating the left-hand side of inequality (2.3) by parts, we have

$$\int_{a}^{b} x^{-r}Fp(x)x^{2v}dx = \frac{b^{-r+2v+1}}{r+2v+1} Fp(b) - \frac{p(2v+1)}{r-2v-1} \int_{a}^{b} x^{-r}Fp(x)x^{2v}dx$$

$$+ \frac{p}{r-2v-1} \int_{a}^{b} x^{-r+1} \frac{[Tu(x)]'}{Tu(x)} Fp(x)x^{2v}dx$$

$$+ \frac{p}{r-2v-1} \int_{a}^{b} x^{-r} [Tv(x)] F^{p-1}(x)x^{2v}dx.$$  

Using the fact that $r > 2v + 1$, $F(b) \geq 0$ and assumption on $\lambda$, we have

$$\int_{a}^{b} x^{-r}Fp(x)\frac{1}{\lambda}x^{2v}dx \leq \int_{a}^{b} x^{-r}Fp(x) \left[ 1 + \frac{p(2v+1)}{r-2v-1} - \frac{p}{r-2v-1} x \frac{[Tu(x)]'}{Tu(x)} \right] x^{2v}dx$$

$$= \frac{b^{-r+2v+1}}{r+2v+1} Fp(b) + \frac{p}{r-2v-1} \int_{a}^{b} x^{-r}F^{p-1}(x)[Tv(x)]x^{2v}dx$$

$$\leq \frac{p}{r-2v-1} \int_{a}^{b} x^{-r}F^{p-1}(x)[Tv(x)]x^{2v}dx.$$  

By Hölder’s inequality

$$\int_{a}^{b} x^{-r}Fp(x)x^{2v}dx$$
Using the fact that

$$
\int_a^b x^{-r} [Tv(x)]^p x^{2\nu} dx
$$

From this, it follows that

$$
\int_a^b x^{-r} F_p(x) x^{2\nu} dx \leq \left( \frac{\lambda p}{r-2\nu-1} \right)^\frac{1}{p} \left[ \int_a^b x^{-r} F^{(p-1)/q}(x) x^{2\nu} dx \right]^\frac{1}{q}.
$$

This proves the theorem.

**Theorem 2.4.** Let $p > q > 0$, $\alpha \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$ and $r > 2\nu + 1$. Let $Tu : (0, \infty) \to (0, \infty)$ be absolutely continuous and let $Tv : [0, \infty) \to [0, \infty)$ be integrable so that

$$
1 + \frac{p}{q} \frac{2\nu}{r-2\nu-1} + \frac{1}{q} \frac{\alpha x [Tu(x)]'}{[Tu(x)]^r} \geq \frac{1}{\lambda} > 0
$$

almost everywhere for some $\lambda > 0$. For any $a \in (0, \infty)$, if

$$
F(x) := \frac{1}{x^{2\nu} [Tu(x)]^\alpha} \int_a^x \frac{Tu(t) Tv(t)}{t} t^{2\nu} dt, \quad x \in (0, \infty)
$$

then for $b \geq a$, we have

$$
\text{(2.4) } \int_a^b x^{-r} F_p^\frac{\nu}{\alpha} (x) x^{2\nu} dx \leq \left( \frac{p - \lambda}{q - r + 2\nu + 1} \right)^\frac{\nu}{\alpha} \int_a^b x^{-r} F^{(p-1)/q}(x) x^{2\nu} dx.
$$

**Proof.** Integrating the left-hand side of inequality (2.4) by parts, we have

$$
\int_a^b x^{-r} F_p(x) x^{2\nu} dx = \int_a^b \frac{b^{-r+2\nu+1} F_p^\frac{\nu}{\alpha} (b)}{\alpha} + \int_a^b \frac{q - r + 2\nu + 1}{x^{-r+2\nu+1}} x^{-r} F_p^\frac{\nu}{\alpha} (x) x^{2\nu} dx
$$

$$
+ \int_a^b \alpha \frac{x^{-r+1}}{q - r + 2\nu + 1} F_p^\frac{\nu}{\alpha} (x) \frac{[Tv(x)]'}{[Tu(x)]^r} x^{2\nu} dx
$$

$$
- \int_a^b \frac{p q}{q - r + 2\nu + 1} \frac{[Tv(x)]'}{[Tu(x)]^r} F_p^\frac{\nu}{\alpha} - 1 (x) x^{2\nu} dx
$$

Using the fact that $r > 2\nu + 1$, $F(b) \geq 0$ and assumption on $\lambda$, we have

$$
\int_a^b x^{-r} F_p^\frac{\nu}{\alpha} (x) \frac{1}{\lambda} x^{2\nu} dx
$$

$$
\leq \int_a^b x^{-r} F_p^\frac{\nu}{\alpha} (x) \left[ 1 + \frac{p}{q} \frac{2\nu}{r-2\nu-1} + \frac{1}{q} \frac{\alpha x [Tu(x)]'}{[Tu(x)]^r} \right] x^{2\nu} dx
$$

$$
= \frac{b^{-r+2\nu+1}}{-r+2\nu+1} F_p^\frac{\nu}{\alpha} (b) + \int_a^b \frac{q - r + 2\nu + 1}{x^{-r+2\nu+1}} x^{-r} F_p^\frac{\nu}{\alpha} (x) x^{2\nu} dx
$$

$$
\leq \frac{1}{q - r + 2\nu + 1} \int_a^b x^{-r} \frac{Tv(x)}{[Tu(x)]^r} F_p^\frac{\nu}{\alpha} - 1 (x) x^{2\nu} dx.
$$
Using the application of Hölder’s inequality with indices $\frac{p}{q}$ and $\frac{p}{p-q}$ respectively, we obtain

\[
\int_a^b x^{-r} F^\frac{p}{q} (x) x^{2\nu} \, dx
\]

\[
\leq \frac{p}{q} \lambda \frac{\lambda}{q - r + 2\nu + 1} \left[ \int_a^b x^{-r} \frac{|Tv(x)|^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}} x^{2\nu} \, dx \right]^\frac{q}{r} \left[ \int_a^b x^{-r} F^\frac{p}{q} (x) x^{2\nu} \, dx \right]^\frac{p-q}{r}.
\]

From this, it follows that

\[
\int_a^b x^{-r} F^\frac{p}{q} (x) x^{2\nu} \, dx \leq \left( \frac{p}{q} \lambda \frac{\lambda}{q - r + 2\nu + 1} \right)^\frac{q}{r} \int_a^b x^{-r} \frac{|Tv(x)|^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}} x^{2\nu} \, dx.
\]

This proves the theorem.

**Theorem 2.5.** Let $p > q > 0$, $\alpha \geq 0 \frac{1}{p} + \frac{1}{q} = 1$, $v > 0$ and $r < 2\nu + 1$. Let $Tu : (0, \infty) \to (0, \infty)$ be absolutely continuous and let $Tv : [0, \infty) \to [0, \infty)$ be integrable so that

\[
1 - \frac{p}{q} \frac{2\nu}{r - 2\nu - 1} - \frac{p}{q} \frac{1}{r - 2\nu - 1} \alpha x \frac{|Tu(x)|'}{|Tu(x)|} \geq \frac{1}{\lambda} > 0
\]

a.e for some $\lambda > 0$. For any $a \in (0, \infty)$, if

\[
F(x) := \frac{1}{x^{2\nu}[Tu(x)]^\alpha} \int_a^x Tu(t)Tv(t) t^{2\nu} \, dt, \quad x \in (0, \infty)
\]

then for $b \geq a$, we have

(2.5) \[
\int_a^b x^{-r} F^\frac{p}{q} (x) x^{2\nu} \, dx \leq \left( \frac{p}{q} \frac{\lambda}{q - r + 2\nu + 1} \right)^\frac{q}{r} \int_a^b x^{-r} \frac{|Tv(x)|^{\frac{p}{q}}}{[Tu(x)]^{(\alpha-1)\frac{p}{q}}} x^{2\nu} \, dx.
\]

**Proof.** This is similar to the proof of Theorem 2.4.

**Theorem 2.6.** Let $p > q > 0$, $\alpha \geq 0 \frac{1}{p} + \frac{1}{q} = 1$, $v > 0$ and $r > 2\nu + 1$. Let $Tu : (0, \infty) \to (0, \infty)$ be absolutely continuous and let $Tv : [0, \infty) \to [0, \infty)$ be integrable so that

\[
1 + \frac{p}{q} \frac{2\nu}{r - 2\nu - 1} + \frac{p}{q} \frac{1}{r - 2\nu - 1} \alpha x \frac{|Tu(x)|'}{|Tu(x)|} \geq \frac{1}{\lambda} > 0
\]

a.e for some $\lambda > 0$. For any $a \in (0, \infty)$, if

\[
F(x) := \frac{1}{x^{2\nu+1}[Tu(x)]^\alpha} \int_a^x Tu(t)Tv(t) t^{2\nu} \, dt, \quad x \in (0, \infty)
\]

then for $b \geq a$, we have

(2.6) \[
\int_a^b x^{-r} F^p (x) x^{2\nu} \, dx \leq \left( \frac{p}{q} \frac{\lambda}{q - r + 2\nu + 1} \right)^p \int_a^b x^{-r} \frac{|Tv(x)|^p}{[Tu(x)]^{(\alpha-1)p}} x^{2\nu} \, dx.
\]
Proof. This is similar to the proof of Theorem 2.3 and Theorem 2.4.

References

Received: January 19, 2005