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An Introductory Note on Relative Derivative and Proportionality

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Abstract

This brief note defines formally the "relative derivative" as a measure of proportional sensitivity; we apply the concept to complex variables and integrate it fully with calculus, by a demonstration of an application to the Stokes's theorem. We motivate the interest by drawing researchers' attention to the fundamental nature of proportionalities underlying all fields.

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1 Introduction

Nature does not come with prescribed measuring units; they are artificially made. To wit, even the absolute speed of light c is contained in the fine structure constant $\simeq 137^{-1}$. In fact, all physical relations, being geometrically invariant, have to be independent of coordinate systems; as such, physical constants are disguises of proportionalities. For example, the gravitational constant G involves the proportionality between the gravitational and the inertial masses of a particle; the proportion of a speed v to c is ubiquitous in relativistic dynamics; molecular, atomic, and nuclear compositions are accounted for by the proportionalities of the constituents therein. In the biological and social domains, ecological equilibria, gender ratios, dietary mixes, and currency exchange rates are common examples of proportionalities. In mathematics, homogeneous functions of degree 0, the law of large numbers, and Lagrange multipliers, to name a few, all touch upon proportionalities.

The standard derivatives, carrying units, are not naturally suited for analyzing proportionalities. In this paper we define formally the concept of "relative derivative," which, in its simplest form, is $\frac{a}{b}\frac{dy}{dx}$. This sensitivity measure has existed well over a century (see, e.g., [5]), but lacked a systematic treatment (cf. [1, 3]). Mathematically, all that is involved in relative derivatives is a simple coordinate transformation of scales; for this reason, all the proofs in this note have been omitted. However, conceptually it is more than just scaling; it also removes any unit. Thus, the a and b above are certain positive quantities sharing the same units as x and y, and are intended to be set at the analyst's discretion.

In the following Section 2, we will first define relative derivatives in $\mathbb{C}$ and re-express the Taylor's series, and then proceed to define derivatives relative to a field of scaling units, to yield corollaries to the Taylor's theorem and the Stokes's theorem, all in terms of relative derivatives in $\mathbb{R}^n$. Next we will show how relative derivatives may simplify analyses by an application to the determination of the dynamic stability of a Lotka-Volterra predator-prey equilibrium in Example 3.6; we will also show how relative derivatives may add new perspectives to existing relationships by an application to the Hamilton-Jacobi equation in Example 3.7. Finally in Section 3 we will conclude with a summary remark.

2 Relative Derivative

Definition 2.1 *Define a scaling unit c_z of $z \in \mathbb{C}$ by*

$$c_z := \rho \cdot (|z| + 1) \tag{1}$$

for some $\rho > 0$ such that

$$\overset{\odot}{z} := \frac{z}{c_z} \tag{2}$$

is unit-free.

Definition 2.2 *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be differentiable at z_0 with $f'(z_0)$; define a relative derivative of f at z_0 by*

$$\overset{\odot}{f}'(z_0) := f'(z_0) \cdot \frac{c_{z_0}}{c_{f(z_0)}} \tag{3}$$

for some scaling units c_{z_0} and $c_{f(z_0)}$; further, $\forall k \in \mathbb{N}$ define a relative derivative of f at z_0 of order k by

$$\overset{\odot}{f}^{(k)}(z_0) := f^{(k)}(z_0) \cdot \frac{c_{z_0}^k}{c_{f(z_0)}}. \tag{4}$$

Corollary 2.3 *(to Taylor's series) Let f be analytic everywhere in the interior of a disk: $B(z_0, r)$; then $\forall z \in B(z_0, r)$*

$$\frac{f(z) - f(z_0)}{c_{f(z_0)}} = \sum_{k=1}^{\infty} \frac{1}{k!} \overset{\odot}{f}^{(k)}(z_0) \cdot (z \overset{\odot}{-} z_0)^k. \tag{5}$$

3 Theorems of Taylor and Stokes

Definition 3.1 *Let $n, r \in \mathbb{N}, \mathbb{E} \subset \mathbb{R}^n$ open, and $f \in C^r(\mathbb{E}, \mathbb{R})$; $\forall k \in \{1, \cdots, r\}$ $\forall j \in \{1, \cdots, k\}$ $\forall i_j \in \{1, \cdots, n\}$ adopt the usual notation $D_{i_1 \cdots i_k} f \equiv \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \equiv f^{(k)}_{i_1 \cdots i_k}$. Assign $\forall \mathbf{x} \equiv (x_i)_{i=1}^n \in \mathbb{E}$ a scaling vector $\mathbf{c_x} \equiv (c_{x_i})_{i=1}^n$, i.e., setting a vector field $\mathcal{C} : \mathbb{E} \longrightarrow \mathbb{R}^n_{\mathbf{x}}$ such that $\mathcal{C}(x_1, \cdots, x_n) = (c_{x_1}, \cdots, c_{x_n})$. Define $\forall \mathbf{x} \in \mathbb{E}$*

$$\overset{\odot}{D}_{i_1 \cdots i_k} f(\mathbf{x}) \equiv \overset{\odot}{f}^{(k)}_{i_1 \cdots i_k}(\mathbf{x}) := (D_{i_1 \cdots i_k} f)(\mathbf{x}) \cdot \left(\frac{c_{x_{i_1}} \cdots c_{x_{i_k}}}{c_{f(\mathbf{x})}} \right), \tag{6}$$

a 1-form $d\overset{\odot}{x}_i(\mathbf{x}) := \frac{dx_i}{c_{x_i}}$, and a $k-$tensor

$$T^k_f(\mathbf{x}) := \frac{1}{k!} \sum_{\substack{i_j \in \{1, \cdots, n\} \\ j = 1, \cdots, k}} \overset{\odot}{D}_{i_1 \cdots i_k} f(\mathbf{x}) \cdot d\overset{\odot}{x}_{i_1}(\mathbf{x}) \otimes \cdots \otimes d\overset{\odot}{x}_{i_k}(\mathbf{x}). \tag{7}$$

Corollary 3.2 *(to Taylor's theorem) Let $n, m, r-1 \in \mathbb{N}, \mathbb{E} \subset \mathbb{R}^n$ open, and $\mathbf{f} \in C^r(\mathbb{E}, \mathbb{R}^m)$; then $\forall \{\mathbf{x}, \mathbf{x} + \triangle\mathbf{x}\} \subset \mathbb{E}$ $\forall j = 1, \cdots, m$*

$$\frac{f_j(\mathbf{x} + \triangle\mathbf{x}) - f_j(\mathbf{x})}{c_{f_j(\mathbf{x})}} = \sum_{k=1}^{r-1} T^k_{f_j}(\mathbf{x})(\triangle\mathbf{x})^k + T^r_{f_j}(\mathbf{p}_j)(\triangle\mathbf{x})^r, \tag{8}$$

where $\mathbf{p}_j = \mathbf{x} + t_j \triangle\mathbf{x}$ for some $t_j \in (0, 1)$.

Definition 3.3 *Let $n, k \in \mathbb{N}$ with $n \geqq k$; $\forall k \geqq 2$ let*

$$\boldsymbol{\omega} \equiv \sum_{\substack{i_1 < \cdots < i_{k-1} \\ \in \{1, \cdots, n\}}} \omega_{i_1 \cdots i_{k-1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \tag{9}$$

be a $(k-1)$-form on an open set $\mathbb{E} \subset \mathbb{R}^n$; for $k = 1$, $\boldsymbol{\omega} \equiv f : \mathbb{E} \longrightarrow \mathbb{R}^n$ for some f. Define $\forall \mathbf{x} \in \mathbb{E}$

$$\overset{\odot}{\omega}_{i_1 \cdots i_{k-1}}(\mathbf{x}) := \frac{c_{x_{i_1}} \cdots c_{x_{i_{k-1}}}}{c_{\omega(\mathbf{x})}} \cdot \omega_{i_1 \cdots i_{k-1}}(\mathbf{x}) \tag{10}$$

and

$$\overset{\odot}{\omega} := \sum_{\substack{i_1 < \cdots < i_{k-1} \\ \in \{1, \cdots, n\}}} \overset{\odot}{\omega}_{i_1 \cdots i_{k-1}} d\overset{\odot}{x}_{i_1} \wedge \cdots \wedge d\overset{\odot}{x}_{i_{k-1}}. \tag{11}$$

Corollary 3.4 *(to Stokes's theorem) If $\overset{\odot}{\omega}$ in the above definition is differentiable on $\mathbb{E}$, then $\forall k$-chain R in $\mathbb{E}$*

$$\int_R d\overset{\odot}{\omega} = \int_{\partial R} \overset{\odot}{\omega}. \tag{12}$$

Remark 3.5 *For $n = 1$ in the preceding corollary, we have*

$$\int_{a/c_a}^{b/c_b} \overset{\odot}{f}'(x)\, d\overset{\odot}{x} = \int_{a/c_a}^{b/c_b} \left(\frac{c_x}{c_{f(a)}} \frac{df(x)}{dx} \right) \frac{dx}{c_x} = \frac{f(b) - f(a)}{c_{f(a)}}. \tag{13}$$

Example 3.6 *We apply relative derivatives to an analysis of the dynamic stability of an equilibrium point in a basic predator-prey model due to Lotka and Volterra ([2], p. 428):*

$$\left\{ \begin{array}{c} \dot{x}_1 = x_1 \cdot (b_1 - i_{12} x_2) \\ \dot{x}_2 = x_2 \cdot (-d_2 + i_{21} x_1) \end{array} \right\}, \tag{14}$$

where x_2 and x_1 denote respectively the populations of the predator and prey, with respective death and birth rates $d_2, b_1 > 0$, and $i_{mn} > 0$ measures the degree of the effect of interaction between x_m and x_n on x_m. Then linearization at $x_1^* = \dfrac{d_2}{i_{21}}$ and $x_2^* = \dfrac{b_1}{i_{12}}$ yields:

$$\left[\begin{array}{c} \dot{x}_1 \\ x_2 \end{array} \right] \simeq \left[\begin{array}{cc} 0 & -\left(\dfrac{i_{12}}{i_{21}} \right) d_2 \\ \left(\dfrac{i_{21}}{i_{12}} \right) b_1 & 0 \end{array} \right] \left[\begin{array}{c} x_1 - \overset{*}{x}_1 \\ x_2 - \overset{*}{x}_2 \end{array} \right], \tag{15}$$

or in relative derivatives

$$\begin{bmatrix} \dot{\frac{x_1-\overset{*}{x}_1}{\overset{*}{x}_1}} \\ \frac{x_2-\overset{*}{x}_2}{\overset{*}{x}_2} \end{bmatrix} \simeq \begin{bmatrix} 0 & -b_1 \\ d_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{x_1-\overset{*}{x}_1}{\overset{*}{x}_1} \\ \frac{x_2-\overset{*}{x}_2}{\overset{*}{x}_2} \end{bmatrix}. \tag{16}$$

By comparison we see that the relative-derivative approach has a simpler Jacobian matrix, involving only b_1 and d_2.

Example 3.7 *Consider the Hamilton-Jacobi equation*

$$\frac{\partial u}{\partial t} + H\left(t, \mathbf{x}, \frac{\partial u}{\partial \mathbf{x}}\right) = 0. \tag{17}$$

Then in relative derivatives we have

$$\frac{\frac{\partial u}{\partial t}}{u} + \left(\sum_{i=1}^{n} \left(\frac{x_i}{u}\frac{\partial u}{\partial x_i}\right)\left(\frac{\dot{x}_i}{x_i}\right)\right)\left(\frac{H}{\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\dot{x}_i}\right) = 0. \tag{18}$$

In classical mechanics H represents the total energy and $\sum_{i=1}^{n} \frac{1}{2}\frac{\partial u}{\partial x_i}\dot{x}_i$ is the kenetic energy.

4 Summary Remark

A key feature of contemporary analyses is that of large scales. Relative derivatives, being free from units, are more amenable to computer simulations. Also, as any measurement is subject to error, its accumulated effect through multiple steps on the reliability of the final result is a matter of serious concern (cf. [4]), and relative derivatives lend themselves readily to fractional analyses of the sizes of the errors. Otherwise, we have shown by the above two examples that relative derivatives may simplify relations as well as uncover new relations. Thus, we surmise that many interesting proportional relationships across diverse fields can be brought to light by the use of the relative derivative.

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