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DISTRIBUTION OF THE RATIO OF MAXWELL AND RICE RANDOM VARIABLES

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Abstract. The distributions of the ratio of independent random variables arise in many applied problems. These have been extensively studied by many researchers. In this paper, the distribution of the ratio $\left|\frac{X}{Y}\right|$ has been derived when X and Y are Maxwell and Rice random variables and are distributed independently of each other. The associated pdfs, cdfs, and kth moments have been given.

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1 Introduction

The distributions of the ratio $\left|\frac{X}{Y}\right|$, when X and Y are independent random variables, arise in many applied problems of biology, economics, engineering, genetics, hydrology, medicine, number theory, order statistics, physics, psychology, etc, (see, for example, [4], [6], and [8], among others, and references therein). The distributions of the ratio $\left|\frac{X}{Y}\right|$, when X and Y are independent random variables and come from the same family, have been extensively studied by many researchers, (see, for example, [9], [10], [11], [12], [14], [16], and [17], among others,). In recent years, there has been a great interest in the study of the above kind when X and Y belong to different families, (see, for example, [13], and [15], among others). In this paper, the distributions of the ratio $\left|\frac{X}{Y}\right|$, when X and Y are independent random variables having Maxwell and Rice distributions respectively, have been investigated. The organization of this paper is as follows. Section 2 contains the derivation of the cdf of the ratio $Z = \left|\frac{X}{Y}\right|$. The pdf and kth moment of the RV $Z = \left|\frac{X}{Y}\right|$ have been derived in Sections 3 and 4 respectively. Some concluding remarks are given in Section 5.

The derivations of the cdf, pdf, and kth moment of $Z = \left|\frac{X}{Y}\right|$ involve some special functions, which are defined as follows (see, for example, [1], [5], and [18], among others, for details). The series

$$ {}_pF_q\left(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z\right) = \sum_{k=0}^{\infty} \left\{ \frac{(\alpha_1)_k\, (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k\, (\beta_2)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \right\}, $$

is called a generalized hypergeometric series of order (p, q), where $(\alpha)_k$ and $(\beta)_k$ represent Pochhammer symbols. For $p = 1$ and $q = 2$, we have generalized hypergeometric function ${}_1F_2$ of order $(1, 2)$, given by

$${}_1F_2\left(\alpha_1; \beta_1, \beta_2; z\right) = \sum_{k=0}^{\infty} \left\{ \frac{(\alpha_1)_k\, z^k}{(\beta_1)_k\, (\beta_2)_k\, k!} \right\}.$$ For $p = 2$ and $q = 1$, we have generalized hypergeometric function ${}_2F_1$ of order $(2, 1)$, given by

$${}_2F_1\left(\alpha, \beta; \gamma; z\right) \equiv F\left(\alpha, \beta; \gamma; z\right) \equiv F\left(\beta, \alpha; \gamma; z\right) = \sum_{k=0}^{\infty} \left\{ \frac{(\alpha)_k (\beta)_k\, z^k}{(\gamma)_k\, k!} \right\}.$$ The following series is known as degenerate hypergeometric function or confluent hypergeometric function:

${}_1F_1(\alpha;\beta;z)=\sum_{k=0}^{\infty}\left\{\frac{(\alpha)_k\, z^k}{(\beta)_k\, k!}\right\}$. The confluent hypergeometric function ${}_1F_1(\alpha;\beta;z)$ is a degenerate form of the generalized hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ of order $(2,1)$ which arises as a solution the confluent hypergeometric differential equation. Note that ${}_1F_1(\alpha,\beta;z)=e^z\ {}_1F_1(\beta-\alpha,\beta;-z)$ which is known as Kummer Transformation. Also, we have $F(\alpha,\beta;\gamma;z)=(1-z)^{-\beta}F\left(\beta,\gamma-\alpha;\gamma;\frac{z}{z-1}\right)$. The integrals $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}\,dt$, and $\gamma(\alpha,x)=\int_0^{x}t^{\alpha-1}e^{-t}\,dt,\ \alpha>0$ are called (complete) gamma and incomplete gamma functions respectively, whereas the integral $\Gamma(\alpha,x)=\int_x^{\infty}t^{\alpha-1}e^{-t}\,dt,\ \alpha>0$ is called the complementary incomplete gamma function. For negative values, gamma function can be defined as $\Gamma\left(-n+\frac{1}{2}\right)=\frac{(-1)^n\,2^n\sqrt{\pi}}{1.3.5.\ldots(2n-1)}$, where $n\geq 0$ is an integer, (see, for example, [2], and [3], among others). The function defined

by $B(p,q)=\int_0^1 t^p\,(1-t)^{q-1}dt=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\ p>0,q>0,$ is known as beta function (or Euler's function of the first kind). The error function is defined by $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$, whereas the complementary error, $erfc(x)$, is defined as $erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-u^2}du=1-erf(x)$. The modified Bessel function of first kind, $I_\nu(x)$, for a real number ν, is defined by $I_\nu(x)=\left(\frac{1}{2}x\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}x^2\right)^k}{(k!)\,\Gamma(\nu+k+1)}$, where $\Gamma(.)$ denotes gamma function. Also, in terms of the confluent hypergeometric function ${}_1F_1$, it can be expressed as

$$I_\nu(x)=\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}e^{-x}\ {}_1F_1\left(\frac{1}{2}+\nu,1+2\nu;2x\right).$$

When $\nu=0$, modified Bessel function of first kind, $I_0(x)$, of order 0 is obtained as follows:

$$I_0(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4} x^2\right)^k}{(k!)^2} \tag{1}$$

For $\operatorname{Re}\left(\nu + \frac{1}{2}\right) > 0, \left|\arg(z)\right| < \frac{\pi}{2}$; or $\operatorname{Re}(z) = 0$ and $\nu = 0$, we have the modified Bessel function of second kind, $K_\nu(x)$, of order ν, given by

$$K_\nu(x) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_1^{\infty} e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} \, dt,$$

or, for $\left|\arg(z)\right| < \frac{\pi}{2}$, $\operatorname{Re}(z^2) > 0$, we have $K_\nu(x) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^{\infty} \frac{e^{-t - \frac{z^2}{4t}}}{(t)^{\nu+1}} \, dt$. For non-integer ν, we have $K_\nu(x) = \frac{\pi \left\{I_{-\nu}(x) - I_\nu(x)\right\}}{2\sin(\nu\pi)}$. The function, denoted by $M_{k,m}(x)$ and defined by

$$M_{k,m}(x) = e^{-x/2} \, x^{m+1/2} \; {}_1F_1\left(\frac{1}{2} + m - k; 1 + 2m; x\right)$$

is called Whittaker function. Also, note that $M_{0,\mu}(z) = 2^{2\mu} \, \Gamma(\mu + 1) \sqrt{z} \, I_\mu\left(\frac{z}{2}\right)$.

The following Lemmas will also be needed in our derivations.

Lemma 1 (Gradshteyn and Ryzhik (2000), [5], Equation (3.381.4), Page 317).
For $\operatorname{Re}(\mu) > 0$, and $\operatorname{Re}(\nu) > 0$,

$$\int_0^{\infty} t^{\nu-1} e^{-\mu t} \, dt = \frac{1}{\mu^\nu} \Gamma(\nu).$$

Lemma 2 (Gradshteyn and Ryzhik (2000), [5], Equation (6.455.2), Page 663).
For $\operatorname{Re}(\alpha + \beta) > 0$, $\operatorname{Re}(\beta) > 0$, and $\operatorname{Re}(\mu + \nu) > 0$,

$$\int_0^{\infty} t^{\mu-1} e^{-\beta t} \gamma(\nu, \alpha t) \, dt = \frac{\alpha^\nu \, \Gamma(\mu + \nu)}{\nu (\alpha + \beta)^{\mu+\nu}} \; {}_2F_1\left(1, \; \mu + \nu; \; \nu + 1; \frac{\alpha}{\alpha + \beta}\right)$$

Lemma 3 (Prudnikov et al. (1986), Volume 2, [18], Equation (2.8.5.6), Page 104).
For $\mathrm{Re}(p) > 0, \mathrm{Re}(\alpha) > -1, |\arg(c)| < \frac{\pi}{4}$,

$$\int_0^\infty t^{\alpha-1} e^{-pt^2} \, erf(ct)\, dt = \frac{c}{\sqrt{\pi}\, p^{(\alpha+1)/2}} \Gamma\left(\frac{\alpha+1}{2}\right) {}_2F_1\left(\frac{1}{2}, \frac{\alpha+1}{2}; \frac{3}{2}; \frac{-c^2}{p}\right)$$

Lemma 4 (Prudnikov et al. (1986), Volume 2, [18], Equations (2.10.3.2), Page 150).
For $\mathrm{Re}(\alpha+\nu) > 0, \mathrm{Re}(p) > 0, \mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(c) > 0$,

$$\int_0^\infty x^{\alpha-1} e^{-px} \gamma(\nu, cx) dx = \frac{c^{\nu}\,\Gamma(\alpha+\nu)}{\nu\,(p)^{\alpha+\nu}} \; {}_2F_1\left(\nu,\; \alpha+\nu;\; \nu+1; -\frac{c}{p}\right)$$

Lemma 5 (Gradshteyn and Ryzhik (2000), [5], Equation (6.643.2), Page 720).
For $\mathrm{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0$,

$$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} I_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)} \beta^{-1} e^{\beta^2/2\alpha} \alpha^{-\mu} M_{-\mu,\,\nu}\left(\frac{\beta^2}{\alpha}\right)$$

where $I_\nu(.)$ denotes modified Bessel function of the first kind, and $M_{k,m}(.)$ denotes Whittaker function, (see definition above).

2 Distribution of the Ratio $\left|\frac{X}{Y}\right|$

Let X and Y be Maxwell and Rice random variables respectively, distributed independently of each other and defined as follows.

2.1 Maxwell Distribution

A continuous random variable X is said to have a Maxwell distribution if its pdf $f_X(x)$ and cdf $F_X(x) = P(X \le x)$ are, respectively, given by

$$f_X(y) = \sqrt{\frac{2}{\pi}}\, a^{3/2}\, x^2\, e^{-a\, x^2/2}\ ,\ x > 0,\ a > 0 \qquad (2)$$

and

$$F_X(x) = \frac{2\gamma\left(\frac{3}{2}, \frac{1}{2} a\, x^2\right)}{\sqrt{\pi}} \ , \qquad (3)$$
$$= erf\left(\sqrt{\frac{a}{2}}\, x\right) - \sqrt{\frac{2a}{\pi}}\, x\, e^{-a\, x^2/2}$$

where $\gamma(a,\, x)$ and $erf(x)$ denote incomplete gamma and error functions respectively, (see definition above).

2.2 Rice Distribution

A continuous random variable Y is said to have a Rice distribution if its pdf $f_Y(y)$ is given by

$$f_Y(y) = \frac{y}{\sigma^2} e^{-(y^2 + v^2)/2\sigma^2} I_0\left(\frac{y\, v}{\sigma^2}\right), \quad y > 0,\ \sigma > 0,\ v \geq 0 \qquad (4)$$

where $I_0(y)$ denotes the modified Bessel function of the first kind, (see definition above). For $|v| = 0$, the expression (4) reduces to a Rayleigh distribution. In what follows, we consider the derivation of the distribution of the product $\left|\frac{X}{Y}\right|$, when X and Y are Maxwell and Rice random variables respectively, distributed independently of each other and defined as above. An explicit expression for the cdf of $\left|\frac{X}{Y}\right|$ in terms of the generalized hypergeometric function ${}_2F_1$ has been derived in Theorem 2.1. In Theorem 2.2, another explicit expression for the cdf

of $\left|\frac{X}{Y}\right|$ in terms of the generalized hypergeometric function ${}_2F_1$, and Whittaker function $M_{k,m}$, has been derived.

Theorem 2.1

Suppose X is a Maxwell random variable with pdf $f_X(x)$ as given in (2) and cdf $F_X(x) = P(X \le x)$ given by (3) in terms of the incomplete gamma function. Also, suppose Y is a Rice random variable with pdf $f_Y(y)$ given by (4) in terms of the modified Bessel function of the first kind $I_0(y)$. Then the cdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$F(z) = \left[\frac{4\, a^{3/2}\,\sigma^3\, z^3\, e^{-\nu^2/2\sigma^2}}{3\sqrt{\pi}}\right] \sum_{k=0}^{\infty} \left\{ \frac{\left(\frac{\nu^2}{2\sigma^2}\right)^k \Gamma\left(k+\frac{5}{2}\right)}{(k!)^2} \; {}_2F_1\left(\frac{3}{2}, k+\frac{5}{2}; \frac{5}{2}; \frac{-a z^2}{\sigma^2}\right) \right\}$$

where ${}_2F_1(.)$ denotes the generalized hypergeometric function of order $(2,1)$, (see definition above).

Proof

Using the expressions (3) for cdf of Maxwell random variable X and the expression (4) for pdf of Rice random variable Y, the cdf $F(z) = \Pr\left(\left|\frac{X}{Y}\right| \le z\right)$ can be expressed as

$$F(z) = \Pr(|X| \le z|Y|) = \int_0^{\infty} F_X(zy)\; f_Y(y)\; dy$$

$$= \left[\frac{2\, e^{-\nu^2/2\sigma^2}}{\sqrt{\pi}\;\sigma^2}\right] \int_0^{\infty} y\, e^{-y^2/2\sigma^2}\, \gamma\left(\frac{3}{2}, \frac{1}{2} a z^2 y^2\right) I_0\left(\frac{\nu y}{\sigma^2}\right) dy \qquad (5)$$

where $y > 0,\ z > 0,\ a > 0,\ \sigma > 0,\ \nu \ge 0$. The proof of Theorem 2.1 (i) easily follows by substituting $y^2 = t$, using the Definition (1) of modified Bessel function of first kind, $I_0(x)$, of order 0, and Lemma 4 in the integral (5) above.

Theorem 2.2

Suppose X is a Maxwell random variable with pdf $f_X(x)$ as given in (2) and cdf $F_X(x) = P(X \le x)$ given by (3) in terms of the error function. Also, suppose Y is a Rice random variable with pdf $f_Y(y)$ given by (4) in terms of the modified Bessel function of the first kind $I_0(y)$. Then the cdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$F(z) = \left\{ \frac{2\sqrt{a}\ \sigma\, z\, e^{-\nu^2/2\sigma^2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{3}{2}\right)}{(k!)^2} \left(\frac{\nu^2}{\sigma^2}\right)^k {}_2F_1\left(\frac{1}{2}, k+\frac{3}{2}; \frac{3}{2}; -a\sigma^2 z^2\right) \right\}$$

$$- \frac{\sqrt{2a}\ \sigma^2\, z\, e^{-\frac{\nu^2 (2\, a\, \sigma^2 z^2 + 1)}{4\sigma^2 (a\, \sigma^2 z^2 + 1)}}}{\nu\,(a\sigma^2 z^2 + 1)}\ M_{-1,\,0}\left(\frac{\nu^2}{2\sigma^2\,(a\,\sigma^2\, z^2 + 1)}\right)$$

where ${}_2F_1(.)$ denotes the generalized hypergeometric function of order $(2, 1)$, and $M_{k,m}(.)$ denotes Whittaker function, (see definition above).

Proof

Using the expressions (3) for cdf of Maxwell random variable X and the expression (4) for pdf of Rice random variable Y, the cdf $F(z) = \Pr\left(\left|\frac{X}{Y}\right| \le z\right)$ can be expressed as

$$F(z) = \Pr(|X| \le z|Y|) = \int_0^{\infty} F_X(z\,y)\ f_Y(y)\ dy$$

$$= \left[\frac{e^{-\nu^2/2\sigma^2}}{\sigma^2}\right] \int_0^{\infty} y\, e^{-y^2/\,2\sigma^2} \left\{ erf\left(\sqrt{\frac{a}{2}}\, z\, y\right) - \sqrt{\frac{2a}{\pi}}\, z\, y\, e^{-\frac{a\, z^2\, y^2}{2}} \right\} I_0\left(\frac{\nu\, y}{\sigma^2}\right) dy \qquad (6)$$

where $y > 0,\ z > 0,\ a > 0,\ \sigma > 0,\ \nu \ge 0$. The proof of Theorem 2.2 easily follows by substituting $y^2 = u$, using the Definition (1) of modified Bessel function of the first kind, $I_0(x)$, of order 0, and then using Lemmas 3 and 5 respectively in the integral (6) above.

Corollary 2.1

Using Lemma 2 in the integral (5) above, the cdf of $Z = \left|\frac{X}{Y}\right|$ in Theorem 2.1 can be easily expressed in the equivalent form as

$$F(z) = \left[\frac{4a^{3/2}\,\sigma^3\, z^3\, e^{-\nu^2/2\sigma^2}}{3\sqrt{\pi}} \right]$$

$$\times \sum_{k=0}^{\infty} \left\{ \frac{\left(\frac{\nu^2}{2\sigma^2}\right)^k \Gamma\left(k+\frac{5}{2}\right)\ {}_2F_1\left(1, k+\frac{5}{2}; \frac{5}{2}; \frac{a\,\sigma^2 z^2}{a\,\sigma^2 z^2+1}\right)}{(k!)^2 \left(a\,\sigma^2 z^2+1\right)^{k+\frac{5}{2}}} \right\}$$

Corollary 2.2

Using the definition of Whittaker function,

$$M_{k,m}(x) = e^{-x/2}\, x^{m+1/2}\ {}_1F_1\left(\frac{1}{2}+m-k; 1+2m; x\right),$$

the cdf of $Z = \left|\frac{X}{Y}\right|$ in Theorem 2.2 can be easily expressed in the equivalent form as

$$F(z) = \left(\frac{2\sqrt{a}\ \sigma z e^{-\nu^2/2\sigma^2}}{\sqrt{\pi}} \right) \sum_{k=0}^{\infty} \left\{ \frac{\left(\frac{\nu^2}{\sigma^2}\right)^k \Gamma\left(k+\frac{3}{2}\right)\ {}_2F_1\left(\frac{1}{2}, k+\frac{3}{2}; \frac{3}{2}; -a\sigma^2 z^2\right)}{(k!)^2} \right\}$$

$$- \left\{ \frac{\sqrt{a}\ \sigma z e^{-\frac{\nu^2}{2\sigma^2}}}{(a\sigma^2 z^2+1)^{\frac{3}{2}}}\ {}_1F_1\left(\frac{3}{2}; 1; \frac{\nu^2}{2\sigma^2\,(a\,\sigma^2 z^2+1)}\right) \right\}$$

where ${}_2F_1$ denotes the generalized hypergeometric function of order $(2, 1)$, and ${}_1F_1$ denotes the generalized hypergeometric function of order $(1, 1)$, (see definition above).

3 PDF of the Ratio $Z = \left|\frac{X}{Y}\right|$

In what follows, without loss of generality, for simplicity of computations, this section discusses the derivation of the pdf of the ratio $Z = \left|\frac{X}{Y}\right|$, when X and Y are Rice and Maxwell random variables distributed according to (4) and (2), respectively, and independently of each other. An explicit expression for the pdf of the ratio $Z = \left|\frac{X}{Y}\right|$ in terms of the gamma function has been derived in Theorem 3.1. The expression for the kth moment of RV $Z = \left|\frac{X}{Y}\right|$ in terms of beta function has been derived in Theorem 3.2.

Theorem 3.1

Suppose X and Y are Rice and Maxwell random variables having pdf given by (4) and (2), respectively. Then the pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$f_Z(z) = \left(\frac{a^{\frac{3}{2}} e^{-\nu^2/2\sigma^2}}{\sqrt{\pi}} \right) \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{5}{2}\right) \nu^{2n} z^{2n+1}}{2^{n-2} \sigma^{2n-3} (n!)^2 \left(z^2 + a\sigma^2\right)^{n+\frac{5}{2}}} \tag{7}$$

Proof

The pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$\begin{aligned} f_Z(z) &= \int_0^{\infty} y\, f_X(zy)\, f_Y(y)\, dy \\ &= \left(\sqrt{\frac{2}{\pi}}\, \frac{a^{\frac{3}{2}}}{\sigma^2} e^{-\nu^2/2\sigma^2} z \right) \int_0^{\infty} y^4 e^{-\frac{z^2 y^2}{2\sigma^2} - \frac{a y^2}{2}} I_0\left(\frac{\nu z y}{\sigma^2}\right) dy, \end{aligned} \tag{8}$$

where $y > 0,\ z > 0,\ a > 0,\ \sigma > 0,\ \nu \geq 0$. The proof of Theorem 3.1 easily follows by using the Definition (1) of modified Bessel function of the first kind, $I_0(x)$, of order 0, substituting $y^2 = t$, and then using Lemma 1 in the integral (8) above.

Theorem 3.2

Suppose X and Y are Rice and Maxwell random variables having pdf given by (4) and (2), respectively. Then the pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$f_Z(z) = \left(3\sqrt{2}\, a^{3/2}\, \sigma^2\, e^{-\nu^2/2\sigma^2}\right) \frac{e^{\frac{\nu^2 z^2}{4\sigma^2 (z^2 + a\,\sigma^2)}}}{(z^2 + a\,\sigma^2)} M_{-2,\,0}\left(\frac{\nu^2 z^2}{2\sigma^2 (z^2 + a\,\sigma^2)}\right)$$

where $M_{k,m}(.)$ denotes Whittaker function, (see definition above).

Proof

The pdf of $Z = \left|\frac{X}{Y}\right|$ can be expressed as

$$f_Z(z) = \int_0^\infty y\, f_X(z\,y)\, f_Y(y)\, dy$$

$$= \left(\sqrt{\frac{2}{\pi}}\, \frac{a^{\frac{3}{2}}}{\sigma^2} e^{-\nu^2/2\sigma^2}\, z\right) \int_0^\infty y^4\, e^{-\frac{z^2 y^2}{2\sigma^2} - \frac{a\,y^2}{2}}\, I_0\left(\frac{\nu\, z\, y}{\sigma^2}\right) dy\,, \tag{9}$$

where $y > 0,\ z > 0,\ a > 0,\ \sigma > 0,\ \nu \ge 0$. The proof of Theorem 3.2 easily follows by substituting $y^2 = t$, and then using Lemma 5 in the integral (9) above.

Corollary 3.1

Using the definition $M_{k,m}(x) = e^{-x/2}\, x^{m+1/2}\ {}_1F_1\left(\frac{1}{2} + m - k; 1 + 2m; x\right)$ of Whittaker function, the pdf of $Z = \left|\frac{X}{Y}\right|$ in Theorem 3.2 can be easily expressed in the equivalent form as

$$f_Z(z) = \left(3\, a^{3/2}\, \sigma\, \nu\, e^{-\nu^2/2\sigma^2}\right) \frac{z}{(z^2 + a\,\sigma^2)^{3/2}}\ {}_1F_1\left(\frac{5}{2}; 1; \frac{\nu^2 z^2}{2\sigma^2 (z^2 + a\,\sigma^2)}\right) \tag{10}$$

where ${}_1F_1$ denotes the generalized hypergeometric function of order $(1, 1)$, (see definition above).

4 kth Moment of the RV $Z = \left|\frac{X}{Y}\right|$

Theorem 4.1

If Z is a random variable with pdf given by (7), then its kth moment can be expressed as

$$E\left(Z^{k}\right)=\left(\frac{a^{\frac{k}{2}}\, e^{-\nu^{2}/2\sigma^{2}}}{\sqrt{\pi}}\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{5}{2}\right)\nu^{2n}}{2^{n-1}\,\sigma^{2n-k}\,(n!)^{2}}\,B\left(\frac{2n+k+2}{2},\frac{3-k}{2}\right),\quad -1\le k<3\ ,$$

where $B(p, q), p > 0, q > 0$, denotes Beta function (or Euler's function of the first kind), (see definition above).

Proof

We have

$$E\left(Z^{k}\right)=\left(\frac{a^{\frac{3}{2}}\, e^{-\nu^{2}/2\sigma^{2}}}{\sqrt{\pi}}\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{5}{2}\right)\nu^{2n}}{2^{n-2}\,\sigma^{2n-3}\,(n!)^{2}}\int_{0}^{\infty}z^{k}\,\frac{z^{2n+1}}{\left(z^{2}+a\,\sigma^{2}\right)^{n+\frac{5}{2}}}\,dz \qquad (11)$$

Substituting $z^2 = u$, and using the equation (3.194.3 / page 285) from Gradshteyn and Ryzhik, [5], in (11), the result of Theorem 4.1 easily follows, provided $-1 \le k < 3$. It is evident from Theorem 4.1 that only the moments of order $k = 1$ and $k = 2$ exist and are given by

$$\alpha_{1}=E(Z)=\left(\frac{a^{\frac{1}{2}}\, e^{-\nu^{2}/2\sigma^{2}}}{\sqrt{\pi}}\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{5}{2}\right)\nu^{2n}}{2^{n-1}\,\sigma^{2n-1}\,(n!)^{2}}\,B\left(\frac{2n+3}{2},1\right)\text{, and}$$

$$\alpha_{2}=E\left(Z^{2}\right)=\left(\frac{a\, e^{-\nu^{2}/2\sigma^{2}}}{\sqrt{\pi}}\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{5}{2}\right)\nu^{2n}}{2^{n-1}\,\sigma^{2n-2}\,(n!)^{2}}\,B\left(n+2,\frac{1}{2}\right).$$

Using the above expressions for α_1 and α_2, one can easily determine the variance given by

$$\beta_2 = Var\ Z = \alpha_2 - \alpha_1^{\ 2}.$$

Further, the first negative moment of Z (by taking $k = -1$ in Theorem 4.1) is given by

$$E\left(Z^{-1}\right) = E\left(\frac{1}{Z}\right) = \left(\frac{e^{-\nu^2/2\sigma^2}}{\sqrt{a\ \pi}}\right) \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{5}{2}\right)\nu^{2n}}{2^{n-1}\ \sigma^{2n+1}\left(n!\right)^2} B\left(\frac{2n+1}{2}, 2\right).$$

For a discussion on the existence of the first negative moment of a continuous random variable and its applications, the interested readers are referred to [7, Section 6.9.1, P. 242] and references therein.

Theorem 4.2

If Z is a random variable with pdf given by (10), then its kth moment can be expressed as

$$E\left(Z^k\right) = \left(\frac{3}{2}\nu\sigma^k a^{\frac{k+2}{2}} e^{-\nu^2/2\sigma^2}\right) B\left(\frac{k+2}{2}, \frac{1-k}{2}\right) {}_2F_2\left(\frac{k+2}{2}, \frac{5}{2}; \frac{3}{2}, 1; \frac{\nu^2}{2\sigma^2}\right),$$

where $-2 < k < 1$, and $B\left(p, q\right)$, $p > 0$, $q > 0$, denotes Beta function (or Euler's function of the first kind) and ${}_2F_2$ denotes the generalized hypergeometric function of order $(2, 2)$, (see definition above).

Proof

We have

$$E\left(Z^k\right) = \left(3\ a^{3/2}\ \sigma\nu e^{-\nu^2/2\sigma^2}\right) \int_0^{\infty} z^k \frac{{}_1F_1\left(\frac{5}{2}; 1; \frac{\nu^2 z^2}{2\sigma^2 (z^2 + a\ \sigma^2)}\right)}{(z^2 + a\ \sigma^2)^{3/2}} z\, dz \qquad (12)$$

Substituting $z^2 = u$, and using the equation (2.22.2.2) / page 335) from Prudnikov et al. (1986), Volume 3, [18], in (12), the result of Theorem 4.2 easily follows, provided $-2 < k < 1$. It is evident that only the first negative moment of Z can be obtained from Theorem 4.2 by taking $k = -1$. This is given by

$$E\left(Z^{-1}\right)=E\left(\frac{1}{Z}\right)=\left(\frac{3\nu\, a^{1/2}\, e^{-\nu^2/2\sigma^2}}{2\sigma}\right)B\left(\frac{1}{2},1\right)\,{}_2F_2\left(\frac{1}{2},\frac{5}{2};\frac{3}{2},1;\frac{\nu^2}{2\sigma^2}\right).$$

5 Concluding Remarks

This paper has derived the distribution of the ratio of two independent random variables X and Y, where X has Maxwell and Y has Rice distribution. The pdf and kth moment of the ratio of two variables are also given. The distribution is obtained as a function of hypergeometric and Whittaker functions, where as the pdf has been obtained as a function of gamma, Whittaker, and hypergeometric functions. We hope the findings of the paper will be useful for the practitioners that have been mentioned in Section 1.

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