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Estimation of Diffusion Parameters in Diffusion Processes and Their Asymptotic Normality[1]

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Abstract

We consider the estimation of diffusion coefficient for a class of univariate and a class of bi-variate diffusion processes by quadratic variation and quadratic covariation. The asymptotic behavior of the estimators for fixed sampling interval, infinite sampling frequency is considered. The estimators are proved to be asymptotically normal and the asymptotic variances are derived.

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1 Introduction

We consider the estimation of diffusion coefficient or volatility, σ in the equation

$$dX_t = u(X_t)dt + \sigma dW_t \tag{1}$$

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and the correlation coefficient r in the system of the equations

$$\begin{cases} dX_t &= u_1(X_t, Y_t)dt + \sigma_1 dW_t \\ dY_t &= u_2(X_t, Y_t)dt + \sigma_2 dB_t, \end{cases} \tag{2}$$

where W_t and B_t are standard Brownian motion with correlation coefficient r. Quadratic Variation and Quadratic Covariation are employed to estimate the diffusion coefficients throughout the present paper. We are concerned with the asymptotic results of the sampling schemes where the final time-point of observation is fixed, say T, and the process is observed more and more frequently.

The estimation problem is of interest in finance. Many univariate diffusion processes of interest in finance are in the form

$$dX_t = \mu(X_t) + \sigma g(X_t)dW_t, \tag{3}$$

which can be transformed into equation (1) with the transformation

$$f(x) = \int^x \frac{1}{g(x)}dx.$$

Similarly, many multivariate diffusion processes of interest in finance can be transformed into equation (2).

Diffusion coefficients, such as volatility or correlation coefficient, are important in pricing derivatives. For example, in Black-Scholes option pricing formula, the option price is closely related with the volatility σ but not with the drift. To price an exchange option, we need σ_1 and σ_2, the volatilities of the two assets, and the correlation coefficient r. The availability of high frequency financial data makes it possible to estimate the parameters more and more accurately and the asymptotic behavior should be considered in order to assess the performance improvement of the estimator using high frequency data.

Many previous work are concerned with the estimation of the volatility σ in univariate diffusion where the drift coefficient is continuous, and even bounded. We are going to consider the estimation not only of the diffusion coefficient σ but also of the correlation coefficient r. Moreover, we allow the drift coefficient to have finite number of discontinuities.

2 Estimators and Their Asymptotic Behaviors

In this section, we will consider the asymptotic behavior of the Quadratic Variation Estimator (QVE) in equations (1) and (2). We denote the Quadratic Variation Estimator of parameter θ by $\tilde{\theta}$.

2.1 Asymptotic Normality for the Quadratic Variation Estimator of σ^2

To establish the asymptotic normality of the estimator $\tilde{\sigma^2}$ in equation (1), we consider first the estimator in the equation where the drift coefficient u is zero. This result will be gradually extended to the case where u is an arbitrary function with countably many discontinuities.

Theorem 1 *Let X_t be a scaled Brownian motion $dX_t = \sigma dW_t$. Let $\Gamma = \{0, \Delta, 2\Delta, ..., n\Delta = T\}$ be a partition on $[0, T]$. We can estimate σ^2 with discrete observation of X_t using quadratic variation. We have the asymptotic normality of the estimator $\tilde{\sigma^2} = \frac{1}{T}\sum(X_{i\Delta} - X_{(i-1)\Delta})^2$: $\frac{1}{\sqrt{\Delta}}(\tilde{\sigma^2} - \sigma^2) \xrightarrow{D} N(0, \frac{2\sigma^4}{T})$ as $\Delta \to 0$.*

Proof. It's a result of Central Limit Theorem.

Let $Y_i = \dfrac{1}{\Delta}(X_{i\Delta} - X_{(i-1)\Delta})^2$, which is a scaled Chi-square distribution.

$E(Y_i) = \sigma^2$; $Var(Y_i) = \dfrac{1}{\Delta^2}(\sigma^4\Delta^2) \cdot 2 = 2\sigma^4$; $Y_i's$ are $i.i.d.$ random variables.

By Central Limit Theorem, let $T = \Delta \cdot n$,

$$\frac{\sum Y_i - \dfrac{T}{\Delta}\sigma^2}{\sqrt{\dfrac{T}{\Delta}}\sqrt{2}\sigma^2} \xrightarrow{D} N(0,1).$$

i.e. $$\frac{\sum(X_{i\Delta} - X_{(i-1)\Delta})^2 - T\sigma^2}{\sqrt{2T\Delta}\sigma^2} \xrightarrow{D} N(0,1).$$

or $$\frac{1}{\sqrt{\Delta}}(\tilde{\sigma^2} - \sigma^2) \xrightarrow{D} N(0, \frac{2\sigma^4}{T}).\diamondsuit$$

Since the drift does not contribute to quadratic variation, it is reasonable to expect that we will have similar asymptotic normality as in Theorem (1) if the drift coefficient $u(x)$ is not too "big". As we show in the following theorem, if u is bounded with countably many discontinuities, then the drift does not contribute to the asymptotic distribution and we have the same asymptotic normality as we have in Theorem (1).

Proposition 1 *Let $dX_t = u(X_t)dt + \sigma dW_t$ have a strong solution and $u(x)$ be bounded with countably many discontinuities. $\Gamma = \{0 = t_0 < t_1 < ...... < t_k =$*

$T\}$ be a partition on $[0,T]$. Then $\frac{1}{|\Gamma|^{\frac{1}{2}}}[\sum(X_{t_i}-X_{t_{i-1}})^2-\sigma^2\sum(W_{t_i}-W_{t_{i-1}})^2] \xrightarrow{P}$ 0 *as* $|\Gamma| \to 0$.

Proof. Assume that $|u(x)| \le M$

$$X_{t_i} - X_{t_{i-1}} = \int_{t_{i-1}}^{t_i} u(X_t)dt + \sigma(W_{t_i} - W_{t_{i-1}}).$$

$$\sum(X_{t_i} - X_{t_{i-1}})^2 = \sigma^2 \sum(W_{t_i} - W_{t_{i-1}})^2 + \sum(\int_{t_{i-1}}^{t_i} u(X_t)dt)^2 + 2\sigma \sum \int_{t_{i-1}}^{t_i} u(X_t)dt(W_{t_i} - W_{t_{i-1}}).$$

$$\begin{aligned}
& \frac{1}{\sqrt{|\Gamma|}} \sum(\int_{t_{i-1}}^{t_i} u(X_t)dt)^2 \\
\le\ & \frac{1}{\sqrt{|\Gamma|}} \sum(t_i - t_{i-1})^2 M^2 \\
\le\ & \frac{1}{\sqrt{|\Gamma|}} |\Gamma| \sum(t_i - t_{i-1}) M^2 \\
\le\ & \sqrt{|\Gamma|} T M^2 \to 0 \text{ as } |\Gamma| \to 0. \\
& \sum \int_{t_{i-1}}^{t_i} u(X_t)dt(W_{t_i} - W_{t_{i-1}}) \\
=\ & \sum \int_{t_{i-1}}^{t_i} [u(X_{t_{i-1}}) + (u(X_t) - u(X_{t_{i-1}}))]dt(W_{t_i} - W_{t_{i-1}}) \\
=\ & \sum \int_{t_{i-1}}^{t_i} u(X_{t_{i-1}})dt(W_{t_i} - W_{t_{i-1}}) + \sum \int_{t_{i-1}}^{t_i} [(u(X_t) - u(X_{t_{i-1}}))]dt(W_{t_i} - W_{t_{i-1}})
\end{aligned}$$

First, consider the contribution of the first term

$$\sum \int_{t_{i-1}}^{t_i} u(X_{t_{i-1}})dt(W_{t_i} - W_{t_{i-1}}).$$

$$\begin{aligned}
& \frac{1}{\sqrt{|\Gamma|}} E[\sum \int_{t_{i-1}}^{t_i} u(X_{t_{i-1}})dt(W_{t_i} - W_{t_{i-1}})]^2 \\
=\ & \frac{1}{\sqrt{|\Gamma|}} E[\sum(t_i - t_{i-1})^2 u^2(X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2] \\
\le\ & E[\sum(t_i - t_{i-1})\sqrt{t_i - t_{i-1}} u^2(X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2]
\end{aligned}$$

$$\begin{aligned}
&= \sum E[E((t_i - t_{i-1})\sqrt{t_i - t_{i-1}}u^2(X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2|\mathbf{F_{t_{i-1}}})] \\
&= \sum (t_i - t_{i-1})^{\frac{5}{2}} E(u^2(X_{t_{i-1}})) \\
&\leq |\Gamma|^{\frac{3}{2}} \sum (t_i - t_{i-1}) E(u^2(X_{t_{i-1}})) \\
&\to 0.
\end{aligned}$$

That is,

$$\frac{1}{\sqrt{|\Gamma|}} \sum \int_{t_{i-1}}^{t_i} u(X_t)dt(W_{t_i} - W_{t_{i-1}}) \xrightarrow{L^2} 0$$

and

$$\frac{1}{\sqrt{|\Gamma|}} \sum \int_{t_{i-1}}^{t_i} u(X_t)dt(W_{t_i} - W_{t_{i-1}}) \xrightarrow{P} 0.$$

Secondly, we consider the contribution from the second term

$$\sum \int_{t_{i-1}}^{t_i} [(u(X_t) - u(X_{t_{i-1}}))]dt(W_{t_i} - W_{t_{i-1}}).$$

We consider the contribution from discontinuous points and continuous point separately. Since u has only countably many discontinuities, $u(x)$ has only countably many discontinuous points on $[0, T]$. We re-partition the interval $[0, T]$ as follows. Let $(s_1, s_2], (s_3, s_4], ..., (s_{2i-1}, s_{2i}]......$ contain the discontinuous points such that

$$s_{2i} - s_{2i-1} \leq (\frac{\epsilon}{i^2 M})^2$$

and let $(v_1, v_2], (v_3, v_4], ..., (v_{2j-1}, v_{2j}]$ cover the rest of the interval $[0, T]$. Let

$$\begin{aligned}
&Q_i(\omega) = \sup_{s_{2i-1} \leq t \leq s_{2i}} |u(X_{s_{2i}}(\omega)) - u(X_{s_{2i-1}}(\omega))| \leq 2M; \\
&\sum \frac{1}{\sqrt{|\Gamma|}} E[\int_{s_{2i-1}}^{s_{2i}} [(u(X_t) - u(X_{s_{2i-1}}))]dt(W_{s_{2i}} - W_{s_{2i-1}})] \\
\leq\ & \sum \frac{1}{\sqrt{|\Gamma|}} E[|W_{s_{2i}} - W_{s_{2i-1}}|Q_i(\omega)(s_{2i} - s_{2i-1})] \\
\leq\ & \sum \frac{1}{\sqrt{|\Gamma|}} (s_{2i} - s_{2i-1})^{\frac{3}{2}} \cdot 2M \\
\leq\ & 2M \sum (s_{2i} - s_{2i-1})^{\frac{1}{2}} \\
\leq\ & 2M \sum \frac{\epsilon}{i^2 M} \\
=\ & \sum \frac{2\epsilon}{i^2}
\end{aligned}$$

Moreover,

$$\sum_{i=1}^{\infty} \frac{2\epsilon}{i^2} \leq c_1 \epsilon$$

for some constant c_1; the sum goes to zero as $\epsilon \to 0$. Now, consider the contribution from all continuous intervals: let $R(\omega) = \sup_i \sup_{v_{i-1} \leq t \leq v_i} |u(X_{v_i}(\omega)) - u(X_{v_{i-1}}(\omega))|$.

$R(\omega) \to 0$ a.s. as $|\Gamma| \to 0$;
$|R(\omega)| \leq 2M(u$ is bounded);
By Bounded Convergence theorem, $E[R(\omega)]^2 \to 0$ as $|\Gamma| \to 0$.

$$\begin{aligned}
& E[\frac{1}{\sqrt{|\Gamma|}} \sum (v_i - v_{i-1}) R(\omega) |W_{v_i} - W_{v_{i-1}}|] \\
= \; & \frac{1}{\sqrt{|\Gamma|}} E[\sum (v_i - v_{i-1}) R(\omega) |W_{v_i} - W_{v_{i-1}}|] \\
\leq \; & E[\sum (v_i - v_{i-1}) (E[R(\omega)]^2)^{\frac{1}{2}}] \\
\leq \; & TE([R(\omega)]^2)^{\frac{1}{2}} \to 0 \text{ as } |\Gamma| \to 0 \diamondsuit
\end{aligned}$$

Next, we will prove a similar asymptotic normality result for a general drift function $u(X_t)$ with countably many discontinuities. The proof is based on the proof of asymptotic normality when the drift is bounded.

Theorem 2 *Let $dX_t = u(X_t)dt + \sigma dW_t$ have a strong solution and $u(x)$ be a function with countably many discontinuities.* $\Gamma = \{0 = t_0 < t_1 < ...... < t_k = T\}$ *be a partition on* $[0, T]$. *Then* $\frac{1}{|\Gamma|^{\frac{1}{2}}}[\sum (X_{t_i} - X_{t_{i-1}})^2 - \sigma^2 \sum (W_{t_i} - W_{t_{i-1}})^2] \xrightarrow{P} 0$.

Proof. First, define $x \wedge M$ as a function of x: $x \wedge M =$

$$\begin{cases} M & if \quad x > M \\ x & if \quad -M \leq x \leq M \\ -M & if \quad x < -M \end{cases}$$

From the proof of Proposition 1, we have

$$G_M(\Gamma) = \frac{1}{\sqrt{|\Gamma|}} \sum \int_{t_{i-1}}^{t_i} [u(X_t) \wedge M] dt (W_{t_i} - W_{t_{i-1}}) \xrightarrow{P} 0 \text{ as } |\Gamma| \to 0.$$

and

$$H_M(\Gamma) = \frac{1}{\sqrt{|\Gamma|}} \sum (\int_{t_{i-1}}^{t_i} [u(X_t) \wedge M] dt)^2 \xrightarrow{P} 0 \text{ as } |\Gamma| \to 0.$$

In other words, for every $\epsilon > 0, \exists \alpha > 0$ such that $P(G_M(\Gamma) > \delta, H_M(\Gamma) > \delta) < \frac{\epsilon}{2} \ \forall |\Gamma| < \alpha, \delta > 0$.

Moreover, since the sequence, $P(|u(X_t)| > M$ for some $0 \le t \le T)$, is a decreasing sequence of M with a limit zero, $\exists$ (a large enough) M such that $P(|u(X_t)| > M$ for some $0 \le t \le T) \le \frac{\epsilon}{2}$.

Define $G(\Gamma) = G_\infty(\Gamma)$ and $H(\Gamma) = H_\infty(\Gamma)$.
$P(G(\Gamma) > \delta, H(\Gamma) > \delta) \le P(G_M(\Gamma) > \delta, H_M(\Gamma) > \delta) + P(|u(X_t)| > M$ for some $0 \le t \le T) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
In other words, $G(\Gamma) \xrightarrow{P} 0$ and $H(\Gamma) \xrightarrow{P} 0. \diamondsuit$

2.2 Asymptotic Normality for the Quadratic Variation Estimator of the correlation coefficient r

The second part of this section is to establish the asymptotic normality of the estimator $\tilde{r}$ in equation (2). The idea of the proof is pretty similar to that for $\tilde{\sigma^2}$: we consider first zero-drift case and extend to the case where the drift functions are arbitrary functions with countably many discontinuities.

Theorem 3 *Let (X_t, Y_t) be a bivariate diffusion process such that*

$$\begin{cases} dX_t &= \sigma_1 dW_t \\ dY_t &= \sigma_2 dB_t, \end{cases}$$

where W_t and B_t are standard Brownian motions with correlation r. Another way to express the process is

$$\begin{cases} dX_t &= \sigma_1 dW_{1t} \\ dY_t &= r\sigma_2 dW_{1t} + \sqrt{1-r^2}\sigma_2 dW_{2t}, \end{cases}$$

where dW_{1t} and dW_{2t} are independent standard Brownian motions. Let $\Gamma = \{0, \Delta, 2\Delta, ..., n\Delta = T\}$ be a partition on $[0,T]$. We can estimate r with *discrete observation of (X_t, Y_t) using quadratic variations and quadratic co-variation:*

$$\tilde{r} = \frac{\sum (X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta})}{\sqrt{\sum (X_{i\Delta} - X_{(i-1)\Delta})^2 \sum (Y_{i\Delta} - Y_{(i-1)\Delta})^2}}.$$

We have the asymptotic normality of the estimator:

$$\frac{1}{\sqrt{\Delta}}(\tilde{r} - r) \xrightarrow{D} N(0, \frac{1+r^2}{T})$$

as $\Delta \to 0$.

Proof. It's a result of Central Limit Theorem.
First of all, consider the denominator of $\tilde{r}$.
We have

$$\sum (X_{i\Delta} - X_{(i-1)\Delta})^2 \xrightarrow{P} \sigma_1^2 T$$

and

$$\sum (Y_{i\Delta} - Y_{(i-1)\Delta})^2 \xrightarrow{P} \sigma_2^2 T$$

We can express $(X_{i\Delta} - X_{(i-1)\Delta})$ and $(Y_{i\Delta} - Y_{(i-1)\Delta})$ as

$$\begin{aligned} X_{i\Delta} - X_{(i-1)\Delta} &= \sigma_1 \sqrt{\Delta} Z_{1i}, \\ Y_{i\Delta} - Y_{(i-1)\Delta} &= \sigma_2 \sqrt{\Delta} r Z_{1i} + \sigma_2 \sqrt{\Delta} \sqrt{1-r^2} Z_{2i} \end{aligned}$$

where (Z_1, Z_2) is a sequence of standard bivariate normal distribution $N(0, I_2)$.

$$\begin{aligned} & E[(X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta})] \\ = \ & E(\sigma_1\sigma_2\Delta r Z_1^2) \\ = \ & r\Delta\sigma_1\sigma_2. \\ & Var[(X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta})] \\ = \ & E(\sigma_1\sigma_2\Delta r Z_1^2 + \sigma_1\sigma_2\Delta\sqrt{1-r^2} Z_1 Z_2)^2 - (r\Delta\sigma_1\sigma_2)^2 \\ = \ & E[(\sigma_1\sigma_2\Delta r)^2 Z_1^4] + E[(\sigma_1\sigma_2\Delta)^2(1-r^2)] - (r\Delta\sigma_1\sigma_2)^2 \\ = \ & 3(r\Delta\sigma_1\sigma_2)^2 - (r\Delta\sigma_1\sigma_2)^2 + (1-r^2)(\sigma_1\sigma_2\Delta)^2 \\ = \ & 2(r\Delta\sigma_1\sigma_2)^2 + (1-r^2)(\sigma_1\sigma_2\Delta)^2 \\ = \ & (1+r^2)(\sigma_1\sigma_2\Delta)^2. \end{aligned}$$

Let $W_i = \frac{1}{\Delta}[(X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta})]$

$E(W_i) = r\sigma_1\sigma_2$; $Var(W_i) = (1+r^2)\sigma_1^2\sigma_2^2$

By Central Limit Theorem,

$$\frac{\sum W_i - \frac{T}{\Delta} r\sigma_1\sigma_2}{\sqrt{\frac{T}{\Delta}}\sqrt{1+r^2}\sigma_1\sigma_2} \xrightarrow{D} N(0,1)$$

$$\frac{1}{\sqrt{\Delta}}[\sum(X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta}) - rT\sigma_1\sigma_2] \xrightarrow{D} N(0,(1+r^2)T\sigma_1^2\sigma_2^2),$$

$$\frac{1}{\sqrt{\Delta}}[\frac{1}{T}\sum(X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta}) - r\sigma_1\sigma_2] \xrightarrow{D} N(0,\frac{(1+r^2)\sigma_1^2\sigma_2^2}{T}),$$

$$\frac{1}{\sqrt{\Delta}}[\frac{1}{\sigma_1\sigma_2 T}\sum(X_{i\Delta} - X_{(i-1)\Delta})(Y_{i\Delta} - Y_{(i-1)\Delta}) - r] \xrightarrow{D} N(0,\frac{1+r^2}{T}).$$

Moreover, $\frac{1}{T}\sum(X_{i\Delta} - X_{(i-1)\Delta})^2 \xrightarrow{P} \sigma_1^2$; $\frac{1}{T}\sum(Y_{i\Delta} - Y_{(i-1)\Delta})^2 \xrightarrow{P} \sigma_2^2$.

Therefore, we have $\frac{1}{\sqrt{\Delta}}(\tilde{r} - r) \xrightarrow{D} N(0,\frac{1+r^2}{T})$ as $\Delta \to 0$.$\diamondsuit$

Next, we will show that if a bivariate diffusion has a pair of drift functions with countably many discontinuities and constant diffusion coefficients as they are in Theorem 3,we have the same asymptotic normality as we have in Theorem 3.

Theorem 4 *Let*

$$\begin{cases} dX_t &= u_1(X_t,Y_t)dt + \sigma_1 dW_t \\ dY_t &= u_2(X_t,Y_t)dt + \sigma_2 dB_t, \end{cases}$$

have a strong solution and $u_1(x,y), u_2(x,y)$ be functions with countably many discontinuities. Then we have the same asymptotic normality of the estimators for the correlation r as we have in Theorem 3.$\diamondsuit$

The extension of the theorem from zero drift to drift with countably many discontinuities is the same as the extension of the asymptotic normality theorem for σ in univariate diffusion. The proof is also pretty similar. First, we establish a proposition that the drift does not contribute to the asymptotic normality if it is bounded with countably many discontinuities. Then we prove the theorem for general functions with countably many discontinuities.

Proposition 2 *Let*

$$\begin{cases} dX_t &= u_1(X_t,Y_t)dt + \sigma_1 dW_t \\ dY_t &= u_2(X_t,Y_t)dt + \sigma_2 dB_t, \end{cases}$$

have a strong solution and $u_1(x,y)$ and $u_2(x,y)$ be bounded with countably many discontinuities, where W_t and B_t are standard Brownian motions with correlation r. Let $\Gamma = \{0 = t_0 < t_1 < ...... < t_k = T\}$ be a partition on $[0,T]$. Another way to express the process is

$$\begin{cases} dX_t &= u_1(X_t,Y_t)dt + \sigma_1 dW_{1t} \\ dY_t &= u_2(X_t,Y_t)dt + \sigma_2 r dW_{1t} + \sigma_2\sqrt{1-r^2}dW_{2t}, \end{cases}$$

Then

$$\frac{1}{\sqrt{|\Gamma|}}\sum[(X_{t_i}-X_{t_{i-1}})(Y_{t_i}-Y_{t_{i-1}}) - \sigma_1\sigma_2 r(W_{1t_i}-W_{1t_{i-1}})^2$$
$$-\sigma_1\sigma_2\sqrt{1-r^2}(W_{1t_i}-W_{1t_{i-1}})(W_{2t_i}-W_{2t_{i-1}})] \xrightarrow{P} 0.$$

Proof. Let M be the bound of u_1 and u_2, i.e. $|u_i(x,y)| \leq M$ for $i = 1, 2$.

$$\begin{aligned} & (X_{t_i}-X_{t_{i-1}})(Y_{t_i}-Y_{t_{i-1}}) \\ = \; & [\int_{t_{i-1}}^{t_i} u_1(X_t,Y_t)dt + \sigma_1(W_{1t_i}-W_{1t_{i-1}})][\int_{t_{i-1}}^{t_i} u_2(X_t,Y_t)dt + \sigma_2 r(W_{1t_i}-W_{1t_{i-1}}) \\ & +\sigma_2\sqrt{1-r^2}(W_{2t_i}-W_{2t_{i-1}})] \\ = \; & \int_{t_{i-1}}^{t_i} u_1(X_t,Y_t)dt \int_{t_{i-1}}^{t_i} u_2(X_t,Y_t)dt + \sigma_1 \int_{t_{i-1}}^{t_i} u_2(X_t,Y_t)dt(W_{1t_i}-W_{1t_{i-1}}) \\ & +\sigma_2 \int_{t_{i-1}}^{t_i} u_1(X_t,Y_t)dt(B_{t_i}-B_{t_{i-1}}) + \sigma_1\sigma_2 r(W_{1t_i}-W_{1t_{i-1}})^2 \\ & +\sigma_1\sigma_2\sqrt{1-r^2}(W_{1t_i}-W_{1t_{i-1}})(W_{2t_i}-W_{2t_{i-1}}). \end{aligned}$$

$$\begin{aligned} & |\int_{t_{i-1}}^{t_i} u_1(X_t,Y_t)dt \int_{t_{i-1}}^{t_i} u_2(X_t,Y_t)dt| \leq (t_i - t_{i-1})^2 M^2; \\ & \sum(\int_{t_{i-1}}^{t_i} u_1(X_t,Y_t)dt \int_{t_{i-1}}^{t_i} u_2(X_t,Y_t)dt) \leq |\Gamma|\sum(t_i - t_{i-1})M^2 = |\Gamma| T\cdot M^2 \to 0. \\ & \int_{t_{i-1}}^{t_i} u_2(X_t,Y_t)dt(W_{1t_i}-W_{1t_{i-1}}) \\ = \; & \int_{t_{i-1}}^{t_i} [(u_2(X_t,Y_t) - u_2(X_{t_{i-1}},Y_{t_{i-1}})) + u_2(X_{t_{i-1}},Y_{t_{i-1}})]dt(W_{1t_i}-W_{1t_{i-1}}) \\ = \; & (t_i - t_{i-1})u_2(X_{t_{i-1}},Y_{t_{i-1}})(W_{1t_i}-W_{1t_{i-1}}) \\ + \; & \int_{t_{i-1}}^{t_i} [(u_2(X_t,Y_t) - u_2(X_{t_{i-1}},Y_{t_{i-1}}))]dt(W_{1t_i}-W_{1t_{i-1}}). \\ & E[\sum(t_i - t_{i-1})u_2(X_{t_{i-1}},Y_{t_{i-1}})(W_{1t_i}-W_{1t_{i-1}})]^2 \end{aligned}$$

$$
\begin{aligned}
&= E[\sum(t_i - t_{i-1})^2 u_2^2(X_{t_{i-1}}, Y_{t_{i-1}})(W_{1t_i} - W_{1t_{i-1}})^2] \\
&\leq M^2 \sum(t_i - t_{i-1})^2 (W_{1t_i} - W_{1t_{i-1}})^2).
\end{aligned}
$$

(Cross terms vanish since for$i < j$,

$$
\begin{aligned}
& E[(t_i - t_{i-1})(t_j - t_{j-1})u_2(X_{t_{i-1}}, Y_{t_{i-1}}) \\
& u_2(X_{t_{j-1}}, Y_{t_{j-1}})(W_{1t_i} - W_{1t_{i-1}})(W_{1t_j} - W_{1t_{j-1}})] \\
=\ & E\{E[(t_i - t_{i-1})(t_j - t_{j-1})u_2(X_{t_{i-1}}, Y_{t_{i-1}}) \\
& u_2(X_{t_{j-1}}, Y_{t_{j-1}})(W_{1t_i} - W_{1t_{i-1}})(W_{1t_j} - W_{1t_{j-1}})|\mathbf{F_{t_{j-1}}}]\} \\
=\ & 0).
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{\sqrt{|\Gamma|}} E[\sum(t_i - t_{i-1})^2 u_2^2(X_{t_{i-1}}, Y_{t_{i-1}})(W_{1t_i} - W_{1t_{i-1}})^2] \\
\leq\ & \frac{M^2}{\sqrt{|\Gamma|}} E[\sum(t_i - t_{i-1})^2 (t_i - t_{i-1})] \\
\leq\ & M^2 |\Gamma|^{\frac{3}{2}} \sum(t_i - t_{i-1}) \to 0.
\end{aligned}
$$

Next, we show that the contribution of the term

$$
\frac{1}{|\Gamma|} \sum \int_{t_{i-1}}^{t_i} [(u_2(X_t, Y_t) - u_2(X_{t_{i-1}}, Y_{t_{i-1}}))] dt (W_{1t_i} - W_{1t_{i-1}})
$$

is zero. The proof is pretty similar to the proof for asymptotic normality of $\tilde{\sigma^2}$. Since u_2 has only countably many discontinuities, $u_2(X_t, Y_t)$ has at most countably many discontinuities on $[0, T]$. Then we re-partition the interval into those containing discontinuous points and those don't contain any discontinuity: Let $(s_1, s_2], (s_3, s_4], ..., (s_{2i-1}, s_{2i}]......$, contain the discontinuous points such that

$$
s_{2i} - s_{2i-1} \leq (\frac{\epsilon}{i^2 M})^2
$$

and let $(v_1, v_2], (v_3, v_4], ..., (v_{2j-1}, v_{2j}]$ cover the rest of the interval $[0, T]$. Let

$$
\begin{aligned}
& Q_i(\omega) = \sup_{s_{2i-1} \leq t \leq s_{2i}} |u_2((X_{s_{2i}}, Y_{s_{2i}})(\omega)) - u_2((X_{s_{2i-1}}, Y_{s_{2i-1}})(\omega))| \leq 2M; \\
& \sum \frac{1}{\sqrt{|\Gamma|}} E[\int_{s_{2i-1}}^{s_{2i}} [(u_2(X_t, Y_t) - u_2(X_{s_{2i-1}}, Y_{s_{2i-1}}))] dt (W_{1s_{2i}} - W_{1s_{2i-1}})] \\
\leq\ & \sum \frac{1}{\sqrt{|\Gamma|}} E[|W_{s_{2i}} - W_{s_{2i-1}}| Q_i(\omega)(s_{2i} - s_{2i-1})]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum \frac{1}{\sqrt{|\Gamma|}}(s_{2i}-s_{2i-1})^{\frac{3}{2}}\cdot 2M \\
&\leq 2M\sum(s_{2i}-s_{2i-1})^{\frac{1}{2}} \\
&\leq 2M\sum\frac{\epsilon}{i^2M} \\
&= \sum\frac{2\epsilon}{i^2}
\end{aligned}
$$

Moreover,

$$\sum_{i=1}^{\infty}\frac{2\epsilon}{i^2}\leq c_1\epsilon$$

for some constant c_1; the sum goes to zero as $\epsilon\to 0$. Now, consider the contribution from all continuous intervals: let $R(\omega)=\sup_i\sup_{v_{i-1}\leq t\leq v_i}|u_2((X_{v_i},Y_{v_i})(\omega))-u_2((X_{v_{i-1}},Y_{v_{i-1}})(\omega))|$.

$R(\omega)\to 0$ a.s. as $|\Gamma|\to 0$;
$|R(\omega)|\leq 2M(u_2$ is bounded);
By Bounded Convergence theorem, $E[R(\omega)]^2\to 0$ as $|\Gamma|\to 0$.

$$
\begin{aligned}
&E[\frac{1}{\sqrt{|\Gamma|}}\sum(v_i-v_{i-1})R(\omega)|W_{1v_i}-W_{1v_{i-1}}|] \\
&= \frac{1}{\sqrt{|\Gamma|}}E[\sum(v_i-v_{i-1})R(\omega)|W_{1v_i}-W_{1v_{i-1}}|] \\
&\leq E[\sum(v_i-v_{i-1})(E[R(\omega)]^2)^{\frac{1}{2}}] \\
&\leq TE([R(\omega)]^2)^{\frac{1}{2}}\to 0 \text{ as } |\Gamma|\to 0\diamondsuit
\end{aligned}
$$

$$\frac{1}{|\Gamma|}\sum\int_{t_{i-1}}^{t_i}[(u_2(X_t,Y_t)-u_2(X_{t_{i-1}},Y_{t_{i-1}}))]dt(W_{1t_i}-W_{1t_{i-1}})\xrightarrow{P}0.$$

Similarly, we can show that

$$\frac{1}{|\Gamma|}\sum\int_{t_{i-1}}^{t_i}[(u_1(X_t,Y_t)-u_1(X_{t_{i-1}},Y_{t_{i-1}}))]dt(B_{1t_i}-B_{1t_{i-1}})\xrightarrow{P}0,$$

and the proof is finished.

The last thing we will prove in this paper is a similar asymptotic normality result for general drift functions $u_1(X_t,Y_t)$ and $u_2(X_t,Y_t)$ with countably many discontinuities. The proof is based on the proof of asymptotic normality when the drift functions are bounded.

Theorem 5 *Let*

$$\begin{cases} dX_t &= u_1(X_t, Y_t)dt + \sigma_1 dW_t \\ dY_t &= u_2(X_t, Y_t)dt + \sigma_2 dB_t, \end{cases}$$

have a strong solution and $u_1(x, y), u_2(x, y)$ be functions with countably many discontinuities. Then we have the same asymptotic normality of the estimators for the correlation r as we have in Theorem 3.$\diamondsuit$

Proof. The proof is similar to that for Theorem 2. We have proved in Proposition that it is true for bounded function. Then we can prove that the probability that the value of the function is greater than M is very small and we establish the proof.

3 Conclusion

We have established the asymptotic normality for the estimators of two important diffusion coefficients: volatility σ and correlation coefficient r. The diffusion coefficients in the equations are constant, while the drift functions have at most countably many discontinuities. The asymptotic normality is invariant for the same diffusion coefficient and any drift function with countably many discontinuities.

References

[1] B. M. Bibby and M. Sorensen, (1995). Martingale estimation functions for discretely observed diffusion processes, *Bernoulli*, 1, 017-039.

[2] F. Black and M. Scholes, (1973). The valuation of options and corporate liabilities *Journal of Political Economy*, 81, 637-654.

[3] K. C. Chan, G. A. Karolyi, F. A. Longstaff and A. B. Sanders,(1992). An empirical comparison of alternative models of short-term interest rate. *The Journal of Finance*, 1, 343-357.

[4] J. Cox, J. Ingersoll, and S. Ross,(1985). A theory of term structure of interest rates. *Econometrica*, 53(2), 358-407.

[5] D. Dacunha-Castelle and D. Florens-Zmirou, (1986). Estimation of the coefficients of a diffusion from discrete observation. *Stochastics*, 19, 263-284.

[6] G. Dohnal, (1987). On estimating the diffusion coefficient. *Journal of Applied Probability*, 24, 105-114.

[7] D. Florens-Zmirou, (1989). Approximate discrete-time schemes for statistics of diffusion processes. *Statistics*, 20, 547-557.

[8] D. Florens-Zmirou, (1993). On estimating the diffusion coefficient from discrete observations. *Journal of Applied Probability*, 30, 790-804.

[9] B. M. Bibby and M. Sorensen, (1995). Martingale estimation functions for discretely observed diffusion processes, *Bernoulli*, 1, 017-039.

[10] I. Karatzas, and S. Shreve, (1992). *Brownian Motion and Stochastic Calculus*, Springer, 2nd edition.

[11] H. Sorensen, (2002). Estimation of diffusion parameters for discretely observed diffusion processes, *Bernoulli*, 8(4), 491-508.

[12] O. Vasicek,(1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5, 177-178.

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