Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 13, 597 - 609

Generalized Relative J-Divergence Measure and Properties[1]

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Abstract. We have considered one parametric generalization of the non-symmetric relative J-divergence measure. The generalized measure is shown belonging to the Csiszár's f-divergence class. Further, we have derived bounds for the generalized measure in terms of well known divergence measures.

Keywords: Divergence measures; Relative information of type s; Relative J-divergence; Csiszár f-divergence; Information inequalities
Mathematics Subject Classification: 94A17; 26D15

1. Introduction

The divergence measures are commonly used to find appropriate distance or difference between two probability distributions. These measures have been applied in several disciplines like probability distributions, signal processing, pattern recognition, finance, economics etc. A convenient classification to differentiate these measures is to categorize them as (Ferentimos and Papaiopannou [10]): parametric, non-parametric and entropy-type measures of information. Parametric measures of information measure the amount of information about an unknown parameter θ supplied by the data and are functions of

[1]This research is partially supported by the Natural Sciences and Engineering Research Council's Discovery Grant to Pranesh Kumar

θ. The best known measure of this type is Fisher's [11] measure of information. Non-parametric measures give the amount of information supplied by the data for discriminating in favor of a probability distribution f_1 against another f_2, or for measuring the distance or affinity between f_1 and f_2. The Kullback-Leibler [14] measure is the best known in this class. Measures of entropy express the amount of information contained in a distribution, that is, the amount of uncertainty associated with the outcome of an experiment. The classical measures of this type are Shannon's [22] and Rényi's [21] measures. Recently new divergence measures and their relationships with the well known divergence measures are studied in Kumar and Chhina [15], Kumar and Hunter [16] and Kumar and Johnson [17]. In this paper, we aim to discuss non-symmetric relative J-divergence measure and its properties. Let

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \ n \geqslant 2,$$

be the set of all complete finite discrete probability distributions. Through out the paper it is understood that the probability distributions $P, Q \in \Gamma_n$.

Following are some non-symmetric divergence measures.

- χ^2-**Divergence** (Pearson [20])

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 \tag{1}$$

- **Relative Information** (Kullback and Leiber [14])

$$K(P||Q) = \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}) \tag{2}$$

- **Relative J-Divergence** (Dragomir et al. [6])

$$D(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i + q_i}{2q_i}\right) \tag{3}$$

Corresponding to each measure, we can easily define measure which is adjoint of other. For example, $K(Q||P)$ is the adjoint of $K(P||Q)$ and vice versa.

The symmetric versions of above measures are given by

$$\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P) \tag{4}$$

and

$$J(P||Q) = K(P||Q) + K(Q||P) = D(P||Q) + D(Q||P). \tag{5}$$

We can also write

$$D(P||Q) = K\left(\frac{P+Q}{2}||Q\right) + K\left(Q||\frac{P+Q}{2}\right) \tag{6}$$

Dragomir et al. [9] studied the measure (4). We call it [28] by *symmetric chi-square divergence*. The measure (5) is well known Jeffreys-Kullback-Leiber [13], [14] *J-divergence*. More details on some of these divergence measures can be seen in Taneja [24, 25, 29] and in on line book by Taneja [26]. In this paper our aim is to work with one parametric generalization of non-symmetric divergence measures (1)-(3).

2. Generalized Measures of Type s

Measure appearing in (2) can be generalized by introducing a real parameter. This generalization is already known in the literature [19].

- **Relative Information of Type s**

$$\Phi_s(P||Q) = \begin{cases} K_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], & s \neq 0, 1 \\ K(Q||P) = \sum_{i=1}^{n} q_i \ln\left(\frac{q_i}{p_i}\right), & s = 0 \\ K(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right), & s = 1 \end{cases}, \tag{7}$$

for all $s \in \mathbb{R}$.

The measure $\Phi_s(P||Q)$ given in (7) admits the following particular cases:

(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.
(ii) $\Phi_0(P||Q) = K(Q||P)$.
(iii) $\Phi_{1/2}(P||Q) = 4\,[1 - B(P||Q)] = 4\,h(P||Q)$.
(iv) $\Phi_1(P||Q) = K(P||Q)$.
(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

Thus we observe that $\Phi_2(P||Q) = \Phi_{-1}(Q||P)$ and $\Phi_1(P||Q) = \Phi_0(Q||P)$.

The measures $B(P||Q)$ and $h(P||Q)$ appearing in part (iii) are given by

$$B(P||Q) = \sqrt{p_i q_i} \tag{8}$$

and

$$h(P||Q) = 1 - B(P||Q) = \frac{1}{2}\sum_{i=1}^{n}(\sqrt{p_i} - \sqrt{q_i})^2 \tag{9}$$

respectively.

The measure $B(P||Q)$ is famous as Bhattacharyya [1] *coefficient* and the measure $h(P||Q)$ is known as Hellinger [12] *discrimination*. For some studied on the measure (7) refer to Taneja [29].

- **Relative J-Divergence of Type s**

We shall propose one parametric generalization of the *relative J-divergence* measure given by (3). This generalization is given by

$$\zeta_s(P||Q) = \begin{cases} D_s(P||Q) = (s-1)^{-1} \sum_{i=1}^{n} (p_i - q_i) \left(\frac{p_i+q_i}{2q_i}\right)^{s-1}, & s \neq 1 \\ D(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i+q_i}{2q_i}\right), & s = 1 \end{cases} \tag{10}$$

for all $s \in \mathbb{R}$.

The measure (10) admit the following particular cases:

(i) $\zeta_0(P||Q) = \Delta(P||Q)$.
(ii) $\zeta_1(P||Q) = D(P||Q)$.
(iii) $\zeta_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

The expression $\Delta(P||Q)$ appearing in part (i) is the well known *triangular discrimination* and is given by

$$\Delta(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}. \tag{11}$$

The measure *relative information of type s*, $\Phi_s(P||Q)$ contains in particular the classical measures such as: Bhattacharyya *coefficient,* χ^2-*divergence* and Hellingar *discrimination*, while the *relative J-divergences of type s*, $\zeta_s(P||Q)$ yield in particular the *triangular discrimination* and χ^2-*divergence*. Some studies on these measures can be seen in Taneja [28], Kumar and Taneja [18]. Some inequalities among the measures (7) and (10) can be seen in [18].

3. Csiszár f-Divergence and Bounds

In this section, we shall give definition of Csiszár *f-divergence* and some results studied recently.

Given a function$f : [0, \infty) \to \mathbb{R}$, the $f-$divergence measure introduced by Csiszár [2] is given by

$$C_f(P||Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{12}$$

for all $P, Q \in \Gamma_n$.

The following theorem is well known in the literature [2, 3].

Theorem 1. *If the function f is convex and normalized, i.e.,* $f(1) = 0$*, then the Csiszár* $f-$*divergence,* $C_f(P||Q)$ *is nonnegative and convex in the pair of probability distribution* $(P,Q) \in \Gamma_n \times \Gamma_n$.

The following theorems provide bounds on the Csiszár *f-divergence*.

Theorem 2. *(Dragomir [4, 5], Taneja [29]) Let* $P, Q \in \Gamma_n$ *be such that* $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \forall i \in \{1, 2, ..., n\}$*, for some* r *and* R *with* $0 < r \leqslant 1 \leqslant R < \infty$*. Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be differentiable convex and normalized i.e.,* $f(1) = 0$*. Then*

$$0 \leqslant C_f(P||Q) \leqslant E_{C_f}(P||Q) \leqslant A_{C_f}(r,R) \tag{13}$$

and

$$0 \leqslant C_f(P||Q) \leqslant B_{C_f}(r,R) \leqslant A_{C_f}(r,R), \tag{14}$$

where

$$E_{C_f}(P||Q) = \sum_{i=1}^{n} (p_i - q_i) f'(\frac{p_i}{q_i}), \tag{15}$$

$$A_{C_f}(r,R) = \frac{1}{4}(R-r)[f'(R) - f'(r)], \tag{16}$$

and

$$B_{C_f}(r,R) = \frac{(R-1)f(r) + (1-r)f(R)}{R-r} \tag{17}$$

Theorem 3. *(Dragomir et al. [7]). Let* $P, Q \in \Gamma_n$ *be such that* $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$*,* $\forall i \in \{1, 2, ..., n\}$*, for some* r *and* R *with* $0 < r \leqslant 1 \leqslant R < \infty$*. Let* $f : [0, \infty) \to \mathbb{R}$ *be a mapping which is normalized, i.e.,* $f(1) = 0$ *so that* f' *is locally absolutely continuous on* $[r,R]$ *and there exists* α*,* β *satisfying*

$$\alpha \leqslant f''(t) \leqslant \beta, \tag{18}$$

for all $t \in (r,R)$

Then

$$\left| C_f(P||Q) - \frac{1}{2} E_{C_f}(P||Q) \right| \leqslant \frac{1}{8}(\beta - \alpha)\chi^2(P||Q) \tag{19}$$

and

$$\left| C_f(P||Q) - E^*_{C_f}(P||Q) \right| \leqslant \frac{1}{8}(\beta - \alpha)\chi^2(P||Q), \tag{20}$$

where $E_{C_f}(P||Q)$ *is as given by (15) and*

$$E^*_{C_f}(P||Q) = \sum_{i=1}^{n} (p_i - q_i) f' \left(\frac{p_i + q_i}{2q_i} \right). \tag{21}$$

Moreover, we have [6]

$$\chi^2(P||Q) \leqslant (R-1)(1-r) \leqslant \frac{(R-r)^2}{4}. \tag{22}$$

Theorem 4. *(Dragomir et al. [8]). Suppose $f : [r, R] \to \mathbb{R}$ is differentiable and f' is of bounded variation, i.e., $\overset{R}{\underset{r}{V}}(f') = \int_r^R |f''(t)|dt < \infty$. Let the constants r, R satisfy the conditions:*

(i) $0 < r < 1 < R < \infty$;
(ii) $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, *for* $i = 1, 2, ..., n$.

Then

$$\left| C_f(P||Q) - \frac{1}{2} E_{C_f}(P||Q) \right| \leqslant V(P||Q) \overset{R}{\underset{r}{V}}(f') \tag{23}$$

and

$$\left| C_f(P||Q) - E^*_{C_f}(P||Q) \right| \leqslant \frac{1}{2} V(P||Q) \overset{R}{\underset{r}{V}}(f'), \tag{24}$$

where $V(P||Q) = \sum_{i=1}^{n} |p_i - q_i|$ is the well known variational distance.

Moreover, we have [8]

$$V(P||Q) \leqslant \frac{2(R-1)(1-r)}{(R-r)} \leqslant \frac{1}{2}(R-r). \tag{25}$$

Let the function $f(x)$ considered in Theorem 4 be convex in $(0, \infty)$ then $f''(x) \geqslant 0$. This gives

$$\begin{aligned} \overset{R}{\underset{r}{V}}(f') &= \int_r^R |f''(t)|dt = \int_r^R f''(t)dt = f'(R) - f'(r) \\ &= (R-r) Z_{C_f}(r, R) = \frac{4}{R-r} A_{C_f}(r, R). \end{aligned} \tag{26}$$

In this situation the bound (23) can be written as

$$\left| C_f(P||Q) - \frac{1}{2} E_{C_f}(P||Q) \right| \leqslant \frac{4}{R-r} A_{C_f}(r, R) V(P||Q) \tag{27}$$

and the bound (24) as

$$\left| C_f(P||Q) - E^*_{C_f}(P||Q) \right| \leqslant \frac{2}{R-r} A_{C_f}(r, R) V(P||Q), \tag{28}$$

where $A_{C_f}(r, R)$ is as given by (16).

Remark 1. *From now onwards, it is understood that, if there are r, R, then $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, $\forall i \in \{1, 2, ..., n\}$, with $0 < r \leqslant 1 \leqslant R < \infty$ or $0 < r < 1 < R < \infty$ (for the applications of Theorem 4), where $P = (p_1, p_2, ..., p_n) \in \Gamma_n$ and $Q = (q_1, q_2, ..., q_n) \in \Gamma_n$.*

In some particular cases studied below, we shall use the notation

$$(29)\qquad L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0 \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \\ \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}, & p = 0 \end{cases},$$

for all $p \in \mathbb{R}$, $a \neq b$. The measure (29) is famous in the literature by *p-logarithmic power mean* [23]. In particular the expression $L_p^p(a,b)$ is understood as

$$(30)\qquad L_p^p(a,b) = \begin{cases} \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}, & p \neq -1, 0 \\ \frac{\ln b - \ln a}{b-a}, & p = -1 \\ 1, & p = 0 \end{cases},$$

for all $a \neq b$.

4. Relative J-Divergence of Type s

In this section we shall obtain bounds applying the theorems given in section 3 for the generalized measure given by (10).

Let us consider

$$(31)\qquad \xi_s(x) = \begin{cases} (s-1)^{-1}(x-1)\left[\left(\frac{x+1}{2}\right)^{s-1} - 1\right], & s \neq 1 \\ (x-1)\ln\left(\frac{x+1}{2}\right), & s = 1 \end{cases},$$

for all $x > 0$ then $C_f(P||Q) = \zeta_s(P||Q)$, where $\zeta_s(P||Q)$ is as given by (10).

Moreover,

$$(32)\qquad \xi_s'(x) = \begin{cases} \frac{1}{2}(x-1)\left(\frac{x+1}{2}\right)^{s-2} + (s-1)^{-1}\left[\left(\frac{x+1}{2}\right)^{s-1} - 1\right], & s \neq 1 \\ \frac{x-1}{x+1} + \ln\left(\frac{x+1}{2}\right), & s = 1 \end{cases},$$

and

$$(33)\qquad \xi_s''(x) = \left(\frac{x+1}{2}\right)^{s-3}\left[\frac{sx + (4-s)}{4}\right],$$

Thus we have $\xi_s''(x) > 0$ for all $x > 0$ and $0 \leqslant s \leqslant 4$, and hence, $\xi_s(x)$ is convex for all $x > 0$ and $0 \leqslant s \leqslant 4$. Also, we have $\xi_s(1) = 0$. In view of this we can say that the *relative J-divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$ and for all $0 \leqslant s \leqslant 4$.

Based on Theorem 2, we have following theorem.

Theorem 5. *For all $P, Q \in \Gamma_n$, $0 \leqslant s \leqslant 4$, the following bounds on $\zeta_s(P||Q)$ hold:*

$$(34)\qquad 0 \leqslant \zeta_s(P||Q) \leqslant E_{\zeta_s(P||Q)}(P||Q) \leqslant A_{\zeta_s(P||Q)}(r,R),$$

and

$$0 \leqslant \zeta_s(P||Q) \leqslant B_{\zeta_s(P||Q)}(r,R) \leqslant A_{\zeta_s(P||Q)}(r,R), \tag{35}$$

where

$$E_{\zeta_s(P||Q)}(P||Q) = \begin{cases} \zeta_s(P||Q) + \sum_{i=1}^{n} \frac{(p_i-q_i)^2}{p_i+q_i}\left(\frac{p_i+q_i}{2q_i}\right)^{s-1} & s \neq 1 \\ D(P||Q) + \Delta(P||Q), & s = 1 \end{cases}, \tag{36}$$

$$A_{\zeta_s(P||Q)}(r,R) = \frac{1}{4}(R-r)^2 \left\{(2-s)2^{2-s} L_{s-3}^{s-3}(r+1,R+1) + s\, 2^{1-s} L_{s-2}^{s-2}(r+1,R+1)\right\} \tag{37}$$

and

$$B_{\zeta_s(P||Q)}(r,R) = 2^{1-s}(R-1)(1-r) L_{s-2}^{s-2}(r+1,R+1). \tag{38}$$

In view of inequalities (35), we have the following corollary.

Corollary 1. *The following bounds hold:*

$$\begin{aligned} 0 &\leqslant \Delta(P||Q) \\ &\leqslant (R-1)(1-r) L_{-2}^{-2}(r+1,R+1) \\ &\leqslant (R-r)^2 L_{-3}^{-3}(r+1,R+1), \end{aligned} \tag{39}$$

$$\begin{aligned} 0 &\leqslant D(P||Q) \\ &\leqslant (R-1)(1-r) L_{-1}^{-1}(r+1,R+1) \\ &\leqslant \frac{(R-r)^2}{2(R+1)(r+1)} + \frac{1}{4}(R-r)^2 L_{-1}^{-1}(r+1,R+1) \end{aligned} \tag{40}$$

and

$$0 \leqslant \chi^2(P||Q) \leqslant (R-1)(1-r) \leqslant \frac{1}{2}(R-r)^2. \tag{41}$$

Proof. It follows in view of (35) by taking $s=0$, $s=1$ and $s=2$ respectively. □

Theorem 6. *For all $P,Q \in \Gamma_n$, $0 \leqslant s \leqslant 4$, the following bounds hold:*

$$\left|\zeta_s(P||Q) - \frac{1}{2} E_{\zeta_s(P||Q)}(P||Q)\right| \leqslant \frac{1}{8}\, \delta_{\zeta_s}(r,R)\chi^2(P||Q) \tag{42}$$

and

$$\left|\zeta_s(P||Q) - E^*_{\zeta_s(P||Q)}(P||Q)\right| \leqslant \frac{1}{8}\, \delta_{\zeta_s}(r,R)\chi^2(P||Q), \tag{43}$$

where

$$E^*_{\zeta_s(P||Q)}(P||Q) = \begin{cases} (s-1)^{-1}\sum_{i=1}^{n}(p_i-q_i)\left(\frac{p_i+3q_i}{4q_i}\right)^{s-1} \\ \qquad + \sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+3q_i}\left(\frac{p_i+3q_i}{4q_i}\right)^{s-1}, & s\neq 1 \\ \sum_{i=1}^{n}(p_i-q_i)\ln\left(\frac{p_i+3q_i}{4q_i}\right)+\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+3q_i}, & s=1 \end{cases} \tag{44}$$

and

$$\delta_{\zeta_s(P||Q)}(r,R) = \begin{cases} 2^{1-s}\left[\frac{s(r-1)+4}{(r+1)^{3-s}}-\frac{s(R-1)+4}{(R+1)^{3-s}}\right], & 0\leqslant s<2 \\ 2^{1-s}\left[\frac{s(R-1)+4}{(R+1)^{3-s}}-\frac{s(r-1)+4}{(r+1)^{3-s}}\right], & 2<s\leqslant 4 \end{cases}. \tag{45}$$

Proof. In order to calculate α and β given in condition (18), we shall make use of third order derivative of $\xi_s(x)$. For all $x\in(0,\infty)$, we have

$$\xi_s'''(x) = 2^{1-s}(x+1)^{s-4}(s-2)\left[sx+(6-s)\right]. \tag{46}$$

¿From (46), we can say that the function $\xi_s''(x)$ is monotonically decreasing function of $x\in(0,\infty)$ for $0\leqslant s<2$ and is monotonically increasing for $2<s\leqslant 6$, and hence, for all $x\in[r,R]$, we have

$$\beta-\alpha = \delta_{\zeta_s}(r,R) = \begin{cases} 2^{1-s}\left[\frac{s(r-1)+4}{(r+1)^{3-s}}-\frac{s(R-1)+4}{(R+1)^{3-s}}\right], & 0\leqslant s<2 \\ 2^{1-s}\left[\frac{s(R-1)+4}{(R+1)^{3-s}}-\frac{s(r-1)+4}{(r+1)^{3-s}}\right], & 2<s\leqslant 6 \end{cases} \tag{47}$$

Now applying (47) over (19) and (20), we get respectively the inequalities (42) and (43). □

We observe here that the expression (47) is valid for $0\leqslant s\leqslant 6$, but we have taken the range $0\leqslant s\leqslant 4$ in (45) because of the convexity and nonnegativity of the function $\zeta_s(P||Q)$ in this range.

Theorem 7. *For all $P,Q\in\Gamma_n$, $0\leqslant s\leqslant 4$, the following bounds hold:*

$$\left|\zeta_s(P||Q)-\frac{1}{2}E_{\zeta_s(P||Q)}(P||Q)\right| \leqslant V(P||Q)\bigvee_r^R(\xi_s') \tag{48}$$

and

$$\left|\zeta_s(P||Q)-E^*_{\zeta_s(P||Q)}(P||Q)\right| \leqslant \frac{1}{2}V(P||Q)\bigvee_r^R(\xi_s'), \tag{49}$$

where

$$\overset{R}{\underset{r}{V}}(\xi_s') = \int_r^R |\xi_s''(t)|dt = \int_r^R \xi_s''(t)dt = f'(R) - f'(r)$$
$$= (R-r)\left\{(2-s)2^{2-s}L_{s-3}^{s-3}(r+1,R+1) + s\,2^{1-s}L_{s-2}^{s-2}(r+1,R+1)\right\} \tag{50}$$

Proof. It follows from Theorem 4. □

In view of inequalities (42) and (48), we have the following corollary.

Corollary 2. *The following bounds hold:*

$$\left|\Delta(P||Q) - 2\sum_{i=1}^{n} q_i \left(\frac{p_i - q_i}{p_i + q_i}\right)^2\right|$$
$$\leqslant \min\left\{2\left[\frac{1}{(r+1)^3} - \frac{1}{(R+1)^3}\right]\chi^2(P||Q), 16(R-r)L_{-3}^{-3}(r+1,R+1)V(P||Q)\right\} \tag{51}$$
$$\leqslant (R-r)^2 \min\left\{\frac{1}{2}\left[\frac{1}{(r+1)^3} - \frac{1}{(R+1)^3}\right], 8\,L_{-3}^{-3}(r+1,R+1)\right\} \tag{52}$$

and

$$|D(P||Q) - \Delta(P||Q)|$$
$$\leqslant \min\left\{\frac{1}{4}\left[\frac{r+3}{(r+1)^2} - \frac{R+3}{(R+1)^2}\right]\chi^2(P||Q), (R-r)\left[\frac{4}{(r+1)(R+1)} + 2\,L_{-1}^{-1}(r+1,R+1)\right]V(P||Q)\right\} \tag{53}$$
$$\leqslant (R-r)^2 \min\left\{\frac{1}{16}\left[\frac{r+3}{(r+1)^2} - \frac{R+3}{(R+1)^2}\right], \frac{2}{(r+1)(R+1)} + L_{-1}^{-1}(r+1,R+1)\right\}. \tag{54}$$

Proof. Inequalities (51) and (53) follows from (42) and (48) respectively by taking $s = 0$ and $s = 1$. Applying (22) and (25) over (51) and (53) we get we get (52) and (54) respectively. □

In view of inequalities (43) and (49), we have the following corollary.

Corollary 3. *We have following bounds:*

$$\left|\Delta(P||Q) - \sum_{i=1}^{n}(q_i - p_i)\left(\frac{4q_i}{p_i+3q_i}\right)^2\right| \tag{55}$$
$$\leqslant \min\left\{\left[\frac{1}{(r+1)^3} - \frac{1}{(R+1)^3}\right]\chi^2(P||Q),\ 4(R-r)L_{-3}^{-3}(r+1,R+1)V(P||Q)\right\}$$

$$\leqslant (R-r)^2 \min\left\{\frac{1}{4}\left[\frac{1}{(r+1)^3} - \frac{1}{(R+1)^3}\right],\ 2\,L_{-3}^{-3}(r+1,R+1)\right\} \tag{56}$$

and

$$\left|D(P||Q) - \sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+3q_i} - \sum_{i=1}^{n}(p_i-q_i)\ln\left(\frac{p_i+3q_i}{4q_i}\right)\right| \tag{57}$$
$$\leqslant \min\left\{\frac{1}{8}\left[\frac{r+3}{(r+1)^2} - \frac{R+3}{(R+1)^2}\right]\chi^2(P||Q),\ (R-r)\left[\frac{1}{(r+1)(R+1)} + \frac{1}{2}\,L_{-1}^{-1}(r+1,R+1)\right]V(P||Q)\right\}.$$

$$\leqslant \frac{(R-r)^2}{2}\min\left\{\frac{1}{16}\left[\frac{r+3}{(r+1)^2} - \frac{R+3}{(R+1)^2}\right],\ \frac{1}{(r+1)(R+1)} + \frac{1}{2}L_{-1}^{-1}(r+1,R+1)\right\}. \tag{58}$$

Proof. Inequalities (55) and (57) follows from (43) and (49) respectively by taking $s = 0$ and $s = 1$. Applying (22) and (25) over (55) and (57) we get we get (56) and (58) respectively. □

Acknowledgments

This work has been done during second author's stay with the *"Mathematics Department, College of Science and Management, University of Northern British Columbia, Prince George BC V2N4Z9, Canada"*, for which he is thankful to the above mentioned university for the support and hospitality.

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Received: May 18, 2006