Net ideals in Banach algebras

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Abstract

Let $A$ be a Banach algebra and let $(a_\alpha)_{\alpha \in I}$ be a net in $A$. The net ideals of $(a_\alpha)_{\alpha \in I}$, are some left (right) ideals of $A$ which related to $(a_\alpha)_{\alpha \in I}$ (cf. Eshaghi 2001). In this paper we give some new examples of this type of ideals and we study some properties of ideals of this form.

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Let $A$ be a Banach algebra and let $(a_\alpha)_{\alpha \in I}$ be a net in $A$, as in [3], the following sets are left ideals of $A$:

$I_\ell(a_\alpha) := \{ a \in A ; \lim_{\alpha} aa_\alpha = a \}$,
$K_\ell(a_\alpha) := \{ a \in A ; \lim_{\alpha} aa_\alpha = 0 \}$,
$C_\ell(a_\alpha) := \{ a \in A ; (aa_\alpha) \text{ is convergent} \}$,
$L_\ell(a_\alpha) := \{ \lim_{\alpha} aa_\alpha ; a \in C_\ell(a_\alpha) \}$,
$B_\ell(a_\alpha) := \{ a \in A ; (aa_\alpha) \text{ is bounded} \}$.

We give some examples of ideals of this form.

bf 1.Example. Let $\lim_{\alpha} a_\alpha = 0$, then $I_\ell(a_\alpha) = L_\ell(a_\alpha) = \{0\}$ and $C_\ell(a_\alpha) = B_\ell(a_\alpha) = K_\ell(a_\alpha) = L_\ell(a_\alpha) = A$.

bf 2.Example. Let $A = \ell^1(\mathbb{N}) = \{ f : \mathbb{N} \rightarrow \mathbb{C}, \sum_{n=1}^{\infty} |f(n)| < \infty \}$. By the following product
\[f \cdot g = f(1)g \quad (f, g \in A)\]
and by the following norm
\[\|f\|_1 = \sum_{n \in \mathbb{N}} f(n) \quad (f \in A).\]

Then \(A\) is a Banach algebra. Let \((a_\alpha)_{\alpha \in I}\) be a net in \(A\). If \(\lim a_\alpha = f\), then
\[L_\ell(a_\alpha) = I_\ell(a_\alpha) = \text{Span}\{f\},\]
\[B_\ell(a_\alpha) = C_\ell(a_\alpha) = A,\]
\[L_\ell(a_\alpha) = \{f \cdot a; \ a \in A\} = \{f(1)a; \ a \in A\} = \begin{cases} \{0\} & f(1) = 0 \\ A & f(1) \neq 0. \end{cases}\]

If \((a_\alpha)_{\alpha \in I}\) be bounded and divergent, then
\[B_\ell(a_\alpha) = A,\]
\[C_\ell(a_\alpha) = \{a \in A; \ a(1) = 0\} = K_\ell(a_\alpha),\]
\[L_\ell(a_\alpha) = I_\ell(a_\alpha) = \{0\}.\]

3. Example. Let \(A = C([0, 1])\) and let
\[f_n(x) = n\chi_{[0, \frac{1}{2n}]}(x) + [2(n-1)(1-nx)+1]\chi_{[\frac{1}{2n}, \frac{1}{n}]}(x) + \chi_{[\frac{1}{n}, 1]}(x) \quad (x \in [0, 1]).\]

As in [3; Example 3], if \(f(x) = x\), then \(f \in C_\ell(f_n) \setminus I_\ell(f_n)\), if
\[f(x) = \frac{1}{2}\chi_{[\frac{1}{2}, 1]}(x) + \sum_{k=1}^{\infty} \left( \frac{1}{2^k} \chi_{[\frac{1}{2^{k+1}}, \frac{1}{2^k}]}(x) + \left[ (x - \frac{1}{2^{k+1}})(2^{k+1} - 1) + \frac{1}{2^{k+1}} \right] \chi_{[\frac{1}{2^{k+1}}, \frac{1}{2^k}]}(x) \right), \quad (x \in [0, 1])\]
then \(f \in B_\ell(f_n) \setminus C_\ell(f_n)\) and for \(f(x) = \sqrt{x} \quad (x \in [0, 1])\), we have \(f \in A \setminus B_\ell(f_n)\).

Let for \(k \in \mathbb{N}\), \(g_k : [0, 1] \longrightarrow \mathbb{C}\) define by \(g_k(x) = (2x)^{\frac{k+1}{k}}\), then by Dine theorem, \(\{g_k\}_{k \in \mathbb{N}}\) is uniformly convergent to \(g(x) = 2x\). But, we have
\[M_n^k = \sup_{x \in [0, \frac{1}{2n}]} |g_k(x)| = \left( \frac{1}{n} \right)^{\frac{k+1}{k}}, \quad (n \geq 2, \ k \in \mathbb{N})\]
and
\[\lim_{n \to \infty} nM_n^k = \lim_{n \to \infty} n^{-\frac{1}{k}} = 0 \quad (k \in \mathbb{N}).\]

By [3; Example 3], \(g_k \in I_\ell(f_n)\). On the other hand \(M_n = \sup_{x \in [0, \frac{1}{2n}]} |g(x)| = \frac{1}{n}\), and \(\lim_{n \to \infty} nM_n = 1\) therefore by [3; Example 3], \(g \not\in I_\ell(f_n)\) and \(I_\ell(f_n)\) is not
In [3], the first author has shown that $I_\ell(a_\alpha)$ and $K_\ell(a_\alpha)$ are closed whenever $(a_\alpha)_{\alpha \in I}$ is a bounded net. Now we take a similar result for $B_\ell(a_\alpha)$ and $C_\ell(a_\alpha)$ as follows.

4. Theorem. Let $(a_\alpha)_{\alpha \in I}$ be a bounded net in $\mathcal{A}$, then the left ideals $B_\ell(a_\alpha)$ and $C_\ell(a_\alpha)$ are closed.

**Proof.** Let $\{c_n\}_{n \in \mathbb{N}}$ be a net in $C_\ell(a_\alpha)$ and $c_n \rightarrow c$. We show that $c \in C_\ell(a_\alpha)$, for this we show that $(ca_\alpha)_{\alpha \in I}$ is a Cauchy net (see for example [2; proposition 1.7]). For $\varepsilon > 0$ there exists $K_\varepsilon \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \quad n \geq K_\varepsilon \Rightarrow \|c_n - c\| < \frac{\varepsilon}{3M}.$$ 

We know that $c_{K_\varepsilon} \in C_\ell(a_\alpha)$, then $(c_{K_\varepsilon}a_\alpha)_{\alpha \in I}$ is a Cauchy net, so there exists $\alpha_0 \in I$ such that

$$\forall \alpha, \beta \in I, \quad \alpha, \beta \geq \alpha_0 \Rightarrow \|c_{K_\varepsilon}a_\alpha - c_{K_\varepsilon}a_\beta\| < \frac{\varepsilon}{3}.$$ 

Let now $\alpha, \beta \geq \alpha_0$, then we have

$$\|ca_\alpha - ca_\beta\| \leq \|ca_\alpha - c_{K_\varepsilon}a_\alpha\| + \|c_{K_\varepsilon}a_\alpha - c_{K_\varepsilon}a_\beta\| + \|c_{K_\varepsilon}a_\beta\| \leq \|c - c_{K_\varepsilon}\| \|a_\alpha\| + \frac{\varepsilon}{3} + \|c_{K_\varepsilon}a_\alpha - c\| \|a_\beta\| < \varepsilon.$$ 

Now, we show that $B_\ell(a_\alpha)$ is closed. Let $\{b_n\}_{n \in \mathbb{N}} \subseteq B_\ell(a_\alpha)$ and $b_n \rightarrow b \in \mathcal{A}$, then there exists $N_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \quad n \geq N_0 \Rightarrow \|b_n - b\| < \frac{1}{M}.$$ 

On the other hand $b_{N_0} \in B_\ell(a_\alpha)$, then $(b_{N_0}a_\alpha)_{\alpha \in I}$ is a bounded net. Let $\|b_{N_0}a_\alpha\| < N$ for every $\alpha \in I$. We have

$$\|ba_\alpha\| \leq \|ba_\alpha - b_{N_0}a_\alpha\| + \|b_{N_0}a_\alpha\| \leq \|b - b_{N_0}\| \|a_\alpha\| + \|b_{N_0}a_\alpha\| < 1 + N.$$ 

Let $(e_\alpha)_{\alpha \in I}$ be a bounded net in $\mathcal{A}$, we related the net ideals related to $(e_\alpha)_{\alpha \in I}$, to the net ideals related to the net $(e_\alpha^2)_{\alpha \in I}$.

5. Theorem. Let $\mathcal{A}$ be a Banach algebra and $(e_\alpha)_{\alpha \in I}$ be a bounded net in $\mathcal{A}$, then the following assertions hold:
such that for every $C\subseteq A$ therefore a \in A$. Since $b \in L(e_a)$, then there exists $b \in L(e_a)$ such that $\lim\limits_{\alpha} ae_a = b$. Since $b \in L(e_a) \subseteq C(e_a)$, then
\[
\lim\limits_{\alpha} ae_a^2 = \lim\limits_{\alpha} \left(\lim\limits_{\alpha} ae_a\right) e_a = \lim\limits_{\alpha} be_a \in \mathcal{A},
\]
therefore $a \in C(e_a^2)$.

6. Corollary. Let $\mathcal{A}$ be a nilpotent Banach algebra and $(e_a)_{a \in I}$ be a net in $\mathcal{A}$. Then $I(e_a) = 0$.

**proof.** By [1; Theorem 46.3], $\mathcal{A}$ is nil algebra, therefore there exists $n \in \mathbb{N}$ such that for every $\alpha \in I$, $e_a^n = 0$, then by above theorem, we have
\[
I(e_a) \subseteq I(e_a^2) \subseteq I(e_a^4) \subseteq \cdots \subseteq I(e_a^{2^n}) = I(e_a) = 0.
\]

The following results are propositions 5.12, 5.3 and 5.9 of [1], respectively.

7. Proposition. Let $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$ and let $b \in \mathcal{B}$. Then $\text{Sp}(\mathcal{A}, b) \subseteq \text{Sp}(\mathcal{B}, b) \cup \{0\}$ and $\partial \text{Sp}(\mathcal{B}, b) \subseteq \partial \text{Sp}(\mathcal{A}, b)$.

8. Proposition. Let $a, b \in \mathcal{A}$, then $\text{Sp}(ab) \setminus \{0\} = \text{Sp}(ba) \setminus \{0\}$.

9. Proposition. Let $J$ be a two-sided ideal of an algebra $\mathcal{A}$ and $\bar{a}$ denote the $J$-coset of element $a \in \mathcal{A}$. Then $\text{Sp}(\mathcal{A}/J, \bar{a}) \subseteq \text{Sp}(\mathcal{A}, a)$.

By applying above propositions we have the following lemma.

10. Lemma. Let $\mathcal{A}$ be a Banach algebra and $(e_a)_{a \in I}$ be a bounded net in $\mathcal{A}$ such that $C(e_a) = I(e_a)$. For $a \in \mathcal{A}$, let
\[
S(a) := \left(\mathbb{C} - \{\lambda \in \mathbb{C}; \ a \in L(\lambda e_a - a)\}\right) \cup \{0\},
\]
then for every $a$ in $I(e_a)$, we have
\[ SP(\mathcal{A}, a) \subseteq S_{(e_\alpha)}(a), \]
\[ \partial(S_{(e_\alpha)}(a)) \subseteq \partial(SP(\mathcal{A}, a)). \]

**proof.** We know that \( I_\ell(e_\alpha) \) is closed subalgebra of \( \mathcal{A} \), let \( B \) be the unitization of \( I_\ell(e_\alpha) \), then for \( \lambda \in \mathbb{C} \), we have

\[ \lambda \not\in SP(I_\ell(e_\alpha), a) \iff \lambda \not\in SP(B, (a, 0)) \]
\[ \iff (-a, \lambda) \in Inv(B) \]
\[ \iff \exists b \in I_\ell(e_\alpha); (b, 1)(-a, \lambda) = (0, 1) \]
\[ \iff \exists b \in I_\ell(e_\alpha); -ba + \lambda b = a \]
\[ \iff \exists \lambda \in I_\ell(e_\alpha); \lim_{\alpha} b(\lambda e_\alpha - a) = a \]
\[ \iff a \in L_\ell(\lambda e_\alpha - a) \]

then \( S_{(e_\alpha)}(a) = SP(I_\ell(e_\alpha), a) \cup \{0\} \), by proposition 7 above, we have \( \partial(S_{(e_\alpha)}(a)) \subseteq \partial(SP(\mathcal{A}, a)) \) and \( SP(\mathcal{A}, a) \subseteq S_{(e_\alpha)}(a) \).

By propositions 8 and 9, we have the following corollary.

**11. Corollary.** Let \( \mathcal{A} \) be a non-unital Banach algebra with bounded approximate identity \((e_\alpha)_{\alpha \in \ell}\). Then for every \( a \) in \( \mathcal{A} \) we have:

(i) \( SP(\mathcal{A}, a) = (\mathbb{C} - \{\lambda \in \mathbb{C}; a \in L_\ell(\lambda e_\alpha - a)\}) \cup \{0\}, \)

(ii) For \( a, b \in \mathcal{A} \) and \( \lambda \in \mathbb{C}, \) \( ab \in L_\ell(\lambda e_\alpha - ab) \) if and only if \( ba \in L_\ell(\lambda e_\alpha - ba) \),

(iii) Let \( J \) be a closed two-sided ideal in \( \mathcal{A} \) and let \( a \in \mathcal{A} \) and \( \lambda \in \mathbb{C}, \) then \( \bar{a} \in L_\ell(\lambda \bar{e}_\alpha - \bar{a}) \) if and only if \( a \in L_\ell(\lambda e_\alpha - a) \).

**References**


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