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Net ideals in Banach algebras

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Abstract

Let $\mathcal{A}$ be a Banach algebra and let $(a_\alpha)_{\alpha\in I}$ be a net in $\mathcal{A}$. The net ideals of $(a_\alpha)_{\alpha\in I}$, are some left (right) ideals of $\mathcal{A}$ which related to $(a_\alpha)_{\alpha\in I}$ (cf. Eshaghi 2001). In this paper we give some new examples of this type of ideals and we study some properties of ideals of this form.

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Let $\mathcal{A}$ be a Banach algebra and let $(a_\alpha)_{\alpha\in I}$ be a net in $\mathcal{A}$, as in [3], the following sets are left ideals of $\mathcal{A}$:

$$\begin{aligned}
&I_\ell(a_\alpha) := \{a \in \mathcal{A}\ ;\quad \lim_\alpha aa_\alpha = a\ \},\\
&K_\ell(a_\alpha) := \{a \in \mathcal{A}\ ;\quad \lim_\alpha aa_\alpha = 0\ \},\\
&C_\ell(a_\alpha) := \{a \in \mathcal{A}\ ;\quad (aa_\alpha)\ is\ \ convergent\},\\
&L_\ell(a_\alpha) := \{\lim_\alpha aa_\alpha\ ;\quad a \in C_\ell(a_\alpha)\ \},\\
&B_\ell(a_\alpha) := \{a \in \mathcal{A}\ ; (aa_\alpha) \text{ is bounded }\}.
\end{aligned}$$

We give some examples of ideals of this form.

bf 1.Example. Let $\lim_\alpha a_\alpha = 0$, then $I_\ell(a_\alpha) = L_\ell(a_\alpha) = \{0\}$ and $C_\ell(a_\alpha) = B_\ell(a_\alpha) = K_\ell(a_\alpha) = L_\ell(a_\alpha) = \mathcal{A}$.

bf 2.Example. Let $\mathcal{A} = \ell^1(\mathbb{N}) = \{f | f : \mathbb{N} \to \mathbb{C},\ \sum_{n=1}^{\infty} |f(n)| < \infty\}$. By the following product

$$f \cdot g = f(1)g \qquad (f, g \in \mathcal{A})$$

and by the following norm

$$\|f\|_1 = \sum_{n \in \mathbb{N}} f(n) \qquad (f \in \mathcal{A}).$$

Then $\mathcal{A}$ is a Banach algebra. Let $(a_\alpha)_{\alpha \in I}$ be a net in $\mathcal{A}$. If $\lim_\alpha a_\alpha = f$, then

$$\begin{aligned} &L_\ell(a_\alpha) = I_\ell(a_\alpha) = Span\{f\}, \\ &B_\ell(a_\alpha) = C_\ell(a_\alpha) = \mathcal{A}, \\ &L_\ell(a_\alpha) = \{f \cdot a;\ a \in \mathcal{A}\} = \{f(1)a;\ a \in \mathcal{A}\} = \begin{cases} \{0\} & f(1) = 0 \\ \mathcal{A} & f(1) \neq 0. \end{cases} \end{aligned}$$

If $(a_\alpha)_{\alpha \in I}$ be bounded and divergent, then

$$\begin{aligned} &B_\ell(a_\alpha) = \mathcal{A}, \\ &C_\ell(a_\alpha) = \{a \in \mathcal{A};\ a(1) = 0\} = K_\ell(a_\alpha), \\ &L_\ell(a_\alpha) = I_\ell(a_\alpha) = \{0\}. \end{aligned}$$

3.Example. Let $\mathcal{A} = C([0,1])$ and let

$$f_n(x) = n\chi_{[0,\frac{1}{2n}]}(x) + [2(n-1)(1-nx)+1]\chi_{[\frac{1}{2n},\frac{1}{n}]}(x) + \chi_{[\frac{1}{n},1]}(x) \qquad (x \in [0,1]) .$$

As in [3; Example 3], if $f(x) = x$, then $f \in C_\ell(f_n) \setminus I_\ell(f_n)$, if

$$\begin{aligned} f(x) \;=\; & \frac{1}{2}\chi_{[\frac{1}{2},1]}(x) + \sum_{k=1}^{\infty} \Big(\frac{1}{2^k}\chi_{[\frac{1}{2^{k+1}-1},\frac{1}{2^k}]}(x) \\ & + \Big[(x - \frac{1}{2^{k+1}})(2^{k+1}-1) + \frac{1}{2^{k+1}}\Big]\chi_{[\frac{1}{2^{k+1}},\frac{1}{2^{k+1}-1}]}(x)\Big) , \qquad (x \in [0,1]) \end{aligned}$$

then $f \in B_\ell(f_n) \backslash C_\ell(f_n)$ and for $f(x) = \sqrt{x}$ $(x \in [0,1])$, we have $f \in \mathcal{A} \backslash B_\ell(f_n)$. Let for $k \in \mathbb{N}$, $g_k : [0,1] \longrightarrow \mathbb{C}$ define by $g_k(x) = (2x)^{\frac{k+1}{k}}$, then by Dine theorem, $\{g_k\}_{k \in \mathbb{N}}$ is uniformly convergent to $g(x) = 2x$. But, we have

$$M_n^k = \sup_{x \in [0,\frac{1}{2n}]} |g_k(x)| = (\frac{1}{n})^{\frac{k+1}{k}} , \qquad (n \geq 2 \ ,\ k \in \mathbb{N})$$

and

$$\lim_{n \to \infty} nM_n^k = \lim_{n \to \infty} n^{-\frac{1}{k}} = 0 \qquad (k \in \mathbb{N}).$$

By [3; Example 3], $g_k \in I_\ell(f_n)$. On the other hand $M_n = \sup_{x \in [0,\frac{1}{2n}]} |g(x)| = \dfrac{1}{n}$, and $\lim_{n \to \infty} nM_n = 1$ therefore by [3; Example 3], $g \notin I_\ell(f_n)$ and $I_\ell(f_n)$ is not closed.

In [3], the first author has shown that $I_\ell(a_\alpha)$ and $K_\ell(a_\alpha)$ are closed whenever $(a_\alpha)_{\alpha\in I}$ is a bounded net. Now we take a similar result for $B_\ell(a_\alpha)$ and $C_\ell(a_\alpha)$ as follows.

4.Theorem. Let $(a_\alpha)_{\alpha\in I}$ be a Bounded net in $\mathcal{A}$, then the left ideals $B_\ell(a_\alpha)$, and $C_\ell(a_\alpha)$ are closed.

proof. Let $\{c_n\}_{n\in\mathbb{N}}$ be a net in $C_\ell(a_\alpha)$ and $c_n \longrightarrow c$. We show that $c \in C_\ell(a_\alpha)$, for this we show that $(ca_\alpha)_{\alpha\in I}$ is a Cauchy net (see for example [2; proposition .1.7]). For $\varepsilon > 0$ there exists $K_\varepsilon \in \mathbb{N}$ such that

$$\forall\ n \in \mathbb{N}, \quad n \geq K_\varepsilon \ \Rightarrow\ \|c_n - c\| < \frac{\varepsilon}{3M}.$$

We know that $c_{K_\varepsilon} \in C_\ell(a_\alpha)$, then $(c_{K_\varepsilon} a_\alpha)_{\alpha\in I}$ is a Cauchy net, so there exists $\alpha_0 \in I$ such that

$$\forall\ \alpha, \beta \in I, \quad \alpha, \beta \geq \alpha_0 \ \Longrightarrow\ \|c_{K_\varepsilon} a_\alpha - c_{K_\varepsilon} a_\beta\| < \frac{\varepsilon}{3}\ .$$

Let now $\alpha, \beta \geq \alpha_0$, then we have

$$\begin{aligned} \|ca_\alpha - ca_\beta\| &\leq \|ca_\alpha - c_{K_\varepsilon} a_\alpha\| + \|c_{K_\varepsilon} a_\alpha - c_{K_\varepsilon} a_\beta\| + \|c_{K_\varepsilon} a_\beta\| \\ &\leq \|c - c_{K_\varepsilon}\|\ \|a_\alpha\| + \frac{\varepsilon}{3} + \|c_{K_\varepsilon} a_\alpha - c\|\ \|a_\beta\| \\ &< \varepsilon\ . \end{aligned}$$

Now, we show that $B_\ell(a_\alpha)$ is closed. Let $\{b_n\}_{n\in\mathbb{N}} \subseteq B_\ell(a_\alpha)$ and $b_n \longrightarrow b \in \mathcal{A}$, then there exists $N_0 \in \mathbb{N}$ such that

$$\forall\ n \in \mathbb{N}, \quad n \geq N_0 \ \Rightarrow\ \|b_n - b\| < \frac{1}{M}\ .$$

On the other hand $b_{N_0} \in B_\ell(a_\alpha)$, then $(b_{N_0} a_\alpha)$ is a bounded net. Let $\|b_{N_0} a_\alpha\| < N$ for every $\alpha \in I$. We have

$$\begin{aligned} \|ba_\alpha\| &\leq \|ba_\alpha - b_{N_0} a_\alpha\| + \|b_{N_0} a_\alpha\| \\ &\leq \|b - b_{N_0}\|\ \|a_\alpha\| + \|b_{N_0} a_\alpha\| \\ &< 1 + N\ . \end{aligned}$$

Let $(e_\alpha)_{\alpha\in I}$ be a bounded net in $\mathcal{A}$, we related the net ideals related to $(e_\alpha)_{\alpha\in I}$, to the net ideals related to the net $(e_\alpha^2)_{\alpha\in I}$.

5.Theorem. Let $\mathcal{A}$ be a Banach algebra and $(e_\alpha)_{\alpha\in I}$ be a bounded net in $\mathcal{A}$, then the following assertions hold:

(1) $I_\ell(e_\alpha) \subseteq I_\ell(e_\alpha^2)$,
(2) $K_\ell(e_\alpha) \subseteq K_\ell(e_\alpha^2)$,
(3) $B_\ell(e_\alpha) \subseteq B_\ell(e_\alpha^2)$,
(4) Let $L_\ell(e_\alpha) \subseteq C_\ell(e_\alpha)$, then $C_\ell(e_\alpha) \subseteq C_\ell(e_\alpha^2)$.

proof. To prove (1), let $a \in I_\ell(e_\alpha)$. Then for every α in I, we have

$$\begin{aligned} \|ae_\alpha^2 - a\| &\leq \|ae_\alpha^2 - ae_\alpha\| + \|ae_\alpha - a\| \\ &\leq \|ae_\alpha - a\| \, \|e_\alpha\| + \|ae_\alpha - a\| \ . \end{aligned}$$

Since $(e_\alpha)_{\alpha\in I}$ is bounded, then trivially $K_\ell(e_\alpha) \subseteq K_\ell(e_\alpha^2)$ and $B_\ell(e_\alpha) \subseteq B_\ell(e_\alpha^2)$. For (4), let $L_\ell(e_\alpha) \subseteq C_\ell(e_\alpha)$ and $a \in C_\ell(e_\alpha)$, then there exists $b \in L_\ell(e_\alpha)$ such that $\lim_\alpha ae_\alpha = b$. Since $b \in L_\ell(e_\alpha) \subseteq C_\ell(e_\alpha)$, then

$$\lim_\alpha ae_\alpha^2 = \lim_\alpha \Big(\lim_\alpha ae_\alpha\Big)e_\alpha = \lim_\alpha be_\alpha \in \mathcal{A} \ ,$$

therefore $a \in C_\ell(e_\alpha^2)$.

6.Corollary. Let $\mathcal{A}$ be a nilpotent Banach algebra and $(e_\alpha)_{\alpha\in I}$ be a net in $\mathcal{A}$. Then $I_\ell(e_\alpha) = 0$.

proof. By [1; Theorem 46.3], $\mathcal{A}$ is nil algebra, therefore there exists $n \in \mathbb{N}$ such that for every $\alpha \in I$, $e_\alpha^n = 0$, then by above theorem, we have

$$I_\ell(e_\alpha) \subseteq I_\ell(e_\alpha^2) \subseteq I_\ell(e_\alpha^4) \subseteq \cdots \subseteq I_\ell(e_\alpha^{2^n}) = I_\ell(0) = 0 \ .$$

The following results are propositions 5.12, 5.3 and 5.9 of [1], respectively.

7.Proposition. let $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$ and let $b \in \mathcal{B}$. Then $Sp(\mathcal{A}, b) \subseteq Sp(\mathcal{B}, b) \cup \{0\}$ and $\partial Sp(\mathcal{B}, b) \subseteq \partial Sp(\mathcal{A}, b)$.

8.Proposition. Let $a, b \in \mathcal{A}$, then $Sp(ab) \setminus \{0\} = Sp(ba) \setminus \{0\}$.

9.Proposition. Let J be a two-sided ideal of an algebra $\mathcal{A}$ and $\bar{a}$ denote the J-coset of element $a \in \mathcal{A}$. Then $Sp(\mathcal{A}/J, \bar{a}) \subseteq Sp(\mathcal{A}, a)$.

By applying above propositions we have the following lemma.

10.Lemma. Let $\mathcal{A}$ be a Banach algebra and $(e_\alpha)_{\alpha\in I}$ be a bounded net in $\mathcal{A}$ such that $C_\ell(e_\alpha) = I_\ell(e_\alpha)$. For $a \in \mathcal{A}$, let

$$S_{(e_\alpha)}(a) := \Big(\mathbb{C} - \{\lambda \in \mathbb{C};\ a \in L_\ell(\lambda e_\alpha - a)\}\Big) \cup \{0\} \ ,$$

then for every a in $I_\ell(e_\alpha)$, we have

$SP(\mathcal{A}, a) \subseteq S_{(e_\alpha)}(a)$,
$\partial(S_{(e_\alpha)}(a)) \subseteq \partial(SP(\mathcal{A}, a))$.

proof. We know that $I_\ell(e_\alpha)$ is closed subalgebra of $\mathcal{A}$, let $\mathcal{B}$ be the unitization of $I_\ell(e_\alpha)$, then for $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\lambda \notin SP(I_\ell(e_\alpha), a) &\iff \lambda \notin SP(B, (a, 0)) \\
&\iff (-a, \lambda) \in Inv(B) \\
&\iff \exists b \in I_\ell(e_\alpha);\ (b, 1)(-a, \lambda) = (0, 1) \\
&\iff \exists b \in I_\ell(e_\alpha);\ -ba + \lambda b = a \\
&\iff \exists b \in I_\ell(e_\alpha);\ \lim_\alpha b(\lambda e_\alpha - a) = a \\
&\iff a \in L_\ell(\lambda e_\alpha - a)
\end{aligned}$$

then $S_{(e_\alpha)}(a) = SP(I_\ell(e_\alpha), a) \cup \{0\}$, by proposition 7 above, we have $\partial(S_{(e_\alpha)}(a)) \subseteq \partial(SP(\mathcal{A}, a))$ and $SP(\mathcal{A}, a) \subseteq S_{(e_\alpha)}(a)$.

By propositions 8 and 9, we have the following corollary.

11.Corollary. Let $\mathcal{A}$ be a non-unital Banach algebra with bounded approximate identity $(e_\alpha)_{\alpha \in I}$. Then for every a in $\mathcal{A}$ we have:

(i) $SP(\mathcal{A}, a) = \Big(\mathbb{C} - \{\lambda \in \mathbb{C};\ a \in L_\ell(\lambda e_\alpha - a)\}\Big) \cup \{0\}$,
(ii) For $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, $ab \in L_\ell(\lambda e_\alpha - ab)$ if and only if $ba \in L_\ell(\lambda e_\alpha - ba)$,
(iii) Let J be a closed two-sided ideal in $\mathcal{A}$ and let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, then $\bar{a} \in L_\ell(\lambda \bar{e}_\alpha - \bar{a})$ if and only if $a \in L_\ell(\lambda e_\alpha - a)$.

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