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ON SOME INTEGRODIFFERENTIAL EQUATIONS OF FRACTIONAL ORDERS

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Abstract

In this note the solutions of some integrodifferential equations with fractional orders in a Banach space are considered.Conditions are given which ensure the existence of a resolvent operator for an integrodifferential equation in a Banach space.

Keywords: Fractional integrodifferential equations - Semigroup - Resolvent operators.

Mathematics Subject Classifications: 26A33, 45D05, 45N05

1.Introduction

In this note we shall be concerned with the fractional integrodifferential equation of the form

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} - Ax(t) = \int_0^t B(t-s)x(s)ds + f(t), \qquad t > 0 \tag{1.1}$$

with the initial condition

$$x(0) = x_0 \in X. \tag{1.2}$$

where X is a Banach space , $0 < \alpha \leq 1$, A is a linear closed operator defined on a dense set D(A)in X into X, $[B(t) : 0 \leq t \leq T_1]$ is a family of linear closed operator defined on a dense set in X into X with domain at least D(A) while the function $f : R^+ \to X$ is absolutely continuous.

Without loss of generality we can assume that $x_0 = 0$. It is assumed also that A generates an analytic semigroup Q(t).This condition implies $\|Q(t)\| \leq K$ for $t \geq 0$ and $\|AQ(t)\| \leq K/t$ for $t > 0$, where$\| \ . \ \|$ is the norm in X and K is a positive constant[1-6].It is also assumed that $x_0 \in D(A)$ and B(t)x is strongly continuously differentiable on [0,T] for $x \in D(A)$. It is further assumed that $[T(t)]_{t\geq 0}$ defined by T(t)f(s)= f(t+s) is a C_0 semigroup on $\aleph$ with generator D_s on domain D(D_s) where $\aleph$ is a subspace of the set of bounded uniformly continuous functions on R^+ into X. Let us suppose that $B(t), B^{`}(t) : Y \to D(D_s)$ where Y is the Banach space formed from D(A),the domain of A, endowed with the graph norm $\|y\|_Y = \|Ay\| + \|y\|$.As A and B(t) are closed operators it follows that A and B(t) are in the set of bounded operators from Y to X B(Y,X) for $0 < t \leq T$ assume further that B(t) is continuous on $0 < t \leq T$ into B(Y,X).It is also supposed that $D_sB(t)$,$D_sB^{`}(t)$ is continuous on $[0,\infty)$ into B(Y,$\aleph$)[7].The theory of fractional calculus is essentially based on the integral convolution between$f^*$and the following generalized function ,introduced by Gelfand and Shilov, $\phi_\lambda(t)$=t_+^λ/ $\Gamma(\lambda)$, where λ is a complex number,t_+^λ=$t^\lambda\theta(t)$,$\theta(t)$ being the Heaviside step function,and $\Gamma(\lambda)$ is the gamma function.

Following Gelfand and Shilov we can define the integral of order $\alpha > 0$ by

$$I^\alpha f^*(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f^*(\theta) d\theta. \tag{1.3}$$

If $0 < \alpha \leq 1$, we can define the derivative of order α by

$$\frac{d^\alpha f^*(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f^*(\theta)}{(t-\theta)^\alpha} d\theta, \tag{1.4}$$

(see [8-10]).If $n-1 < \alpha < n$,then it easy to see that

$$\frac{d^\alpha f^*(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{*(n)}(\theta)}{(t-\theta)^{\alpha+1-n}} d\theta + \sum_{k=0}^{n} f^{*(k)}(0^+)\phi_{k-\alpha+1}(t), \tag{1.5}$$

where

$$\frac{d^{\alpha} f^{*}(t)}{dt^{\alpha}}=\frac{d^{n-1}}{dt^{n}} I^{n-\alpha} f^{*}(t)=\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{0}^{t} \frac{f^{*}(\theta)}{(t-\theta)^{\alpha+1-n}} d\theta, \tag{1.6}$$

(comp[11-13])

We shall first consider the fractional evolution equation of the form

$$\frac{d}{dt}(\frac{d^{\alpha} x(t)}{dt^{\alpha}}-Ax(t))=B_{1} x(t)+\int_{0}^{t} B_{2}(t-s) x(s) ds+f(t) \qquad (1.7).$$

with the initial conditions

$$x(0)=0, \qquad \frac{dx(0)}{dt}=0. \tag{1.8}$$

then we obtain the solution of the problem (1.1),(1.2).Finally an application is considered

2.Existence of solutions

Theorem 2.1

Assume that B_1 and $B_2(t)$ are closed linear operators defined on dense sets in X into X with domains at least D(A),and $B_2(t)x$ is strongly continuous in t on [0,T] for $x \in D(A)$.It is also assumed that the domain of $B_2(t)$ does not depend on t.If $B_1, B_2(t) : Y \to D(D_s)$ and $f_1 : R^+ \to X$ is continuous,then the problem (1.7),(1.8) has a unique solution

$$x(t)=\alpha \int_{0}^{t} \int_{0}^{\eta} \int_{0}^{\infty} \theta \zeta_{\alpha}(\theta)(t-\eta)^{\alpha-1} Q((t-\eta)^{\alpha} \theta) R(\eta-v) f(v) d\theta dv d\eta. \tag{2.1}$$

Proof:

First we assume that

$$\frac{d^{\alpha} x(t)}{dt^{\alpha}}-Ax(t)=U(t). \tag{2.2}$$

Hence formally, from [14]:

$$x(t) = \alpha \int_0^t \int_0^\infty \theta \zeta_\alpha(\theta)(t-\eta)^{\alpha-1} Q((t-\eta)^\alpha \theta) U(\eta) d\theta d\eta, \tag{2.3}$$

where $\zeta_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$ .From (1.7), (1.8), (2.2) and (2.3) we get

$$\begin{aligned} \frac{dU(t)}{dt} &= \alpha \int_0^t \int_0^\infty \theta \zeta_\alpha(\theta)(t-\eta)^{\alpha-1} B_1 Q((t-\eta)^\alpha \theta) U(\eta) d\theta d\eta \\ &+ \alpha \int_0^t \int_0^\phi \int_0^\infty \theta \zeta_\alpha(\theta) B_2(t-\phi)(\phi-\eta)^{\alpha-1} Q((\phi-\eta)^\alpha \theta) U(\eta) d\theta d\eta d\phi \\ &+ f_1(t), \end{aligned} \tag{2.4}$$

$$U(0) = 0. \tag{2.5}$$

We rewrite equation (2.4) as

$$\begin{aligned} \frac{dU(t)}{dt} &= \int_0^t B_3(t-\eta) U(\eta) d\eta + \int_0^t B_4(t-\eta) U(\eta) d\eta + f_1(t) \\ &= \int_0^t B_5(t-\eta) U(\eta) d\eta + f_1(t), \end{aligned} \tag{2.6}$$

where

$B_3(t-\eta) = \alpha \int_0^\infty \theta \zeta_\alpha(\theta)(t-\eta)^{\alpha-1} B_1 Q((t-\eta)^\alpha \theta) d\theta,$

$B_4(t-\eta) = \alpha \int_\eta^t \int_0^\infty \theta \zeta_\alpha(\theta) B_2(t-\phi)(\phi-\eta)^{\alpha-1} Q((\phi-\eta)^\alpha \theta) d\theta d\phi,$

$B_5(t-\eta) = B_3(t-\eta) + B_4(t-\eta).$ (2.7)

Equation(2.6)has a unique resolvent operator R(t) which satisfies

$$\frac{\partial R(t-\eta)}{\partial t} = \int_\eta^t B_5(t-r) R(r-\eta) dr. \tag{2.8}$$

Then equations (2.5),(2.6) has a unique solution

$$U(t) = \int_0^t R(t-\eta) f(\eta) d\eta. \tag{2.9}$$

By differentiating (2.9) and using (2.8).Hence by Fubini's theorem we obtain (2.4).

Then

$$x(t) = \alpha \int_0^t \int_0^\eta \int_0^\infty \theta \zeta_\alpha(\theta)(t-\eta)^{\alpha-1} Q((t-\eta)^\alpha \theta) R(\eta - v) f(v) d\theta dv d\eta.$$

Theorem 2.2

The problem (1.1),(1.2) has a unique solution

$$x(t) = \alpha \frac{d}{dt} \int_0^t \int_0^\eta \int_0^\infty \theta\zeta_\alpha(\theta)(t-\eta)^{\alpha-1} Q((t-\eta)^\alpha\theta) R_1(\eta-v) f(v) d\theta dv d\eta. \tag{2.10}$$

Proof:

Let $v(t) = \int_0^t x(\theta)d\theta.$

We can rewrite equation (1.1),(1.2) as

$$\begin{aligned} \frac{d}{dt}\left(\frac{d^\alpha v}{dt^\alpha} - Av(t)\right) &= \int_0^t B(t-s)\frac{dv}{ds}ds + f(t) \\ &= B(0)v(t) + \int_0^t B^{'}(t-s)v(s)ds + f(t), \qquad (2.11) \\ \frac{dv(0)}{dt} &= 0. \qquad (2.12) \end{aligned}$$

By using the previous theorem ,we get

$$v(t) = \alpha \int_0^t \int_0^\eta \int_0^\infty \theta\zeta_\alpha(\theta)(t-\eta)^{\alpha-1} Q((t-\eta)^\alpha\theta) R_1(\eta-v) f(v) d\theta dv d\eta, \tag{2.13}$$

where

$$\begin{aligned} \frac{\partial R_1(t-\eta)}{\partial t} &= \alpha \int_\eta^t \int_0^\infty \theta\zeta_\alpha(\theta) B(0)(t-r)^{\alpha-1} Q((t-r)^\alpha\theta) d\theta dr \\ &+ \alpha \int_\eta^t \int_\eta^\phi \int_0^\infty \theta\zeta_\alpha(\theta) B^{'}(t-\phi)(\phi-r)^{\alpha-1} Q((\phi-r)^\alpha\theta) d\theta d\phi dr. \end{aligned}$$

Hence the required result .

3.Application

As an application we consider the following equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \sum_{|q|\leq\ 2m} a_q(x) D^q u(x,t) = \int_0^t \sum_{|q|\leq\ k} b_q(x,t-s) D^q u(x,s) ds + f(x,t), t > o \tag{3.1}$$

with the initial condition

$$u(x,0) = u_0(x). \tag{3.2}$$

where $x = (x_1, ..., x_n) \in$ S ,S is a bounded domain in the n- dimensional Euclidean space R^n with smooth boundary ∂S,let $L^2(S)$ be the set of all square integrable functions on S.Denote by $C^m(S)$ the set of all continuous functions defined on S,which have continuous partial derivatives of order less than or equal to m,and by $C_0^m(S)$ we denote the set of all functions $f \in C^m(S)$ with compact support. Let $H^m(S)$ be the completion of the space $C^m(S)$ with respect to the norm $\|f\|_m$,

$$\|f\|_m = [\sum_{|q|\leq m} \int_S [D^q f(x)]^2 dx]^{1/2}.$$

By $H_0^m(S)$ we denote the completion of the space $C_0^m(S)$ with respect to the norm $\|f\|_m$. Let A and B be the differential operators defined by

$$A = \sum_{|q|\leq 2m} a_q(x) D^q, \qquad B = \sum_{|q|\leq k} b_q(x,t) D^q.$$

It is assumed that:

(1) The domain of definition D(A) of A is defined by $D(A) = H^{2m}(S) \cap H_0^m$.

(2) $(-1)^{m-1} \sum_{|q|=2m} a_q(x)\xi^q \geq \delta|\xi|^{2m}$, for all $x \in S \cup \partial S$ and for all $\xi = (\xi_1, ..., \xi_n) \neq (0, ...0)$,where δ is a positive constant independent of x and ξ, $(|\xi|^2 = \xi_1^2 + ... + \xi_n^2, \ \ \xi^q = \xi_1^{q_1} .... \xi_n^{q_n})$.

(3) The coefficients a_q are continuous on S ,for all $|q| \leq 2m$ and the coefficients on $S \times [0,T]$,for all $|q| \leq k$,where $T > 0$.

(4) f is continuous on $S \times [0,T]$.

Under these conditions the differential operator A generates an analytic semigroup Q(t),where:

$$Q(t)u_0 = \int_{R^n} G(x-\xi, t) u_0(\xi) d\xi,$$

where $d\xi = d\xi_1.....d\xi_n$ and G is the fundamental solution of the Cauchy problem:

$$\frac{\partial u(x,t)}{\partial t} = \sum_{|q|\le 2m} a_q(x) D^q u(x,t),$$

$$u(x,0) = u_0(x)$$

Now as an application of theorem 2.2, we can prove that the Cauchy problem (3.1),(3.2) has a unique solution.

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