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Primes in the Interval $[2n, 3n]$

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Abstract. Is it true that for all integer $n > 1$ and $k \leq n$ there exists a prime number in the interval $[kn, (k+1)n]$? The case $k = 1$ is the Bertrand's postulate which was proved for the first time by P. L. Chebyshev in 1850, and simplified later by P. Erdős in 1932, see [2]. The present paper deals with the case $k = 2$. A positive answer to the problem for any $k \leq n$ implies a positive answer to the old problem whether there is always a prime in the interval $[n^2, n^2 + n]$, see [1, p. 11].

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1. THE RESULT

Throughout the paper $\ln(x)$ is the logarithm with base e of x and $\pi(x)$ is the number of prime numbers not greater than x. We let n run through the natural numbers and p through the primes.

Lemma 1.1. *The following inequalities hold:*

1. *If n is even then*

$$\binom{\frac{3n}{2}}{n} < \sqrt{6.75}^{\,n}.$$

2. *If n is even such that $n > 152$ then*

$$\binom{\frac{3n}{2}}{n} > \sqrt{6.5}^{\,n}.$$

3. *If n is odd and $n > 7$ then*

$$\binom{\frac{3n+1}{2}}{n} < \sqrt{6.75}^{\,n-1}.$$

4. If $n > 945$ *then*

$$(\frac{6.5}{\sqrt{27}})^n > (3n)^{\frac{\sqrt{3n}}{2}}.$$

Proof. (1,2) By induction on n. We have

$$\binom{3}{2} < 6.75 \text{ and } \binom{\frac{3\cdot 154}{2}}{154} > \sqrt{6.5}^{\ 154}.$$

Assume now that the two inequalities hold for $\binom{3n}{2n}$. Then

$$\begin{aligned}\binom{3n+3}{2n+2} &= \binom{3n}{2n}\frac{(3n+1)(3n+2)(3n+3)}{(n+1)(2n+1)(2n+2)} \\ &= \binom{3n}{2n}\frac{3(3n+1)(3n+2)}{(2n+1)(2n+2)} \\ &= \binom{3n}{2n}\frac{27n^2+27n+6}{4n^2+6n+2}.\end{aligned}$$

It now suffices to note that for all n

$$\frac{27n^2+27n+6}{4n^2+6n+2} < 6.75$$

and for all $n > 12$

$$6.5 < \frac{27n^2+27n+6}{4n^2+6n+2}.$$

(3) By induction on n. We have $\binom{14}{9} < (6.75)^4$. Assume now that the result is true for $\binom{3n+2}{2n+1}$. Then

$$\begin{aligned}\binom{3n+5}{2n+3} &= \binom{3n+2}{2n+1}\frac{3(3n+4)(3n+5)}{2(n+2)(2n+3)} \\ &< (6.75)^n\ 6.75 \\ &= (6.75)^{n+1}.\end{aligned}$$

(4) Note that the following three inequalities are equivalent:

$$(\frac{6.5}{\sqrt{27}})^n > (3n)^{\frac{\sqrt{3n}}{2}}$$

$$n\ln\frac{6.5}{\sqrt{27}} > \frac{\sqrt{3n}}{2}\ln 3n$$

$$\frac{2}{\sqrt{3}}\ln\frac{6.5}{\sqrt{27}} > \frac{\ln 3n}{\sqrt{n}}.$$

Then the result follows since the function $\frac{\ln 3x}{\sqrt{x}}$ is decreasing and

$$\frac{2}{\sqrt{3}}\ln\frac{6.5}{\sqrt{27}} > \frac{\ln(3\cdot 946)}{\sqrt{946}}.$$

□

Lemma 1.2. *1. If n is even then*

$$\prod_{\frac{n}{2}<p\leq\frac{3n}{4}} p \cdot \prod_{n<p\leq\frac{3n}{2}} p < \binom{\frac{3n}{2}}{n}.$$

2. If n is odd then

$$\prod_{\frac{n+1}{2}<p\leq\frac{3n}{4}} p \cdot \prod_{n<p\leq\frac{3n+1}{2}} p < \binom{\frac{3n+1}{2}}{n}.$$

Proof. (1) We have

$$\binom{\frac{3n}{2}}{n} = \frac{(n+1)\cdots\frac{3n}{2}}{\frac{n}{2}!}. \tag{1.1}$$

Then clearly $\prod_{n<p\leq\frac{3n}{2}} p$ divides $\binom{\frac{3n}{2}}{n}$. Furthermore, if $\frac{n}{2} < p \leq \frac{3n}{4}$ then $2p$ occurs in the numerator of (1.1) but p does not occur in the denominator. Then after simplification of $2p$ with an even number from the denominator we get the prime factor p in $\binom{\frac{3n}{2}}{n}$. Thus $\prod_{\frac{n}{2}<p\leq\frac{3n}{4}} p$ divides $\binom{\frac{3n}{2}}{n}$ too and the required inequality follows.
(2) Similar to the first part. □

To prove Bertrand's postulate, P. Erdős needed to check the result for $n = 2, \dots, 113$, refer to [3, p. 173]. Our theorem requires a separate check for $n = 2, \dots, 945$ but we omitted to list them for reasons of space.

Theorem 1.3. *For any positive integer $n > 1$ there is a prime number between $2n$ and $3n$.*

Proof. It can be checked (using Mathematica for instance) that for $n = 2, \dots, 945$ there is always a prime between $2n$ and $3n$. Now let $n > 945$. As

$$\binom{3n}{2n} = \frac{(2n+1)(2n+2)\cdots 3n}{1\cdot 2\cdots n}, \tag{1.2}$$

the product of primes between $2n$ and $3n$, if there are any, divides $\binom{3n}{2n}$. Following the notation used in [3], we let

$$T_1 = \prod_{p\leq\sqrt{3n}} p^{\beta(p)}, \quad T_2 = \prod_{\sqrt{3n}<p\leq 2n} p^{\beta(p)}, \quad T_3 = \prod_{2n+1\leq p\leq 3n} p,$$

such that

$$\binom{3n}{2n} = T_1 T_2 T_3. \tag{1.3}$$

The prime decomposition of $\binom{3n}{2n}$ implies that the powers in T_2 are less than 2, see [3, p. 24] for the prime decomposition of $\binom{n}{j}$. Moreover, we claim that if a prime p satisfies $\frac{3n}{4} < p \leq n$ then its power in T_2 is 0. Clearly, a prime p with this condition appears in the denominator of (1.2) but $2p$ does not, and $3p$ appears in the numerator of (1.2) but $4p$ does not. This way p cancels and the claim follows. Furthermore, if $\frac{3n}{2} < p \leq 2n$ then its power in T_2 is 0 because such a prime p is neither in the denominator nor in the numerator of (1.2) and $2p > 3n$. Now by Lemma 1.2 and the fact that $\prod_{p\leq x} p < 4^x$, refer to [3, p. 167], we have that:

- If n is even then

$$\begin{aligned} T_2 &< \prod_{\sqrt{3n}<p\leq\frac{n}{2}} p \cdot \prod_{\frac{n}{2}<p\leq\frac{3n}{4}} p \cdot \prod_{n<p\leq\frac{3n}{2}} p \\ &< 4^{\frac{n}{2}} \binom{\frac{3n}{2}}{n} \\ &< 4^{\frac{n}{2}} (6.75)^{\frac{n}{2}} \\ &= \sqrt{27}^{\,n}. \end{aligned} \tag{1.4}$$

- If n is odd then

$$\begin{aligned} T_2 &< \prod_{\sqrt{3n}<p\leq\frac{n+1}{2}} p \cdot \prod_{\frac{n+1}{2}<p\leq\frac{3n}{4}} p \cdot \prod_{n<p\leq\frac{3n+1}{2}} p \\ &< 4^{\frac{n+1}{2}} \binom{\frac{3n+1}{2}}{n} \\ &< 4^{\frac{n+1}{2}} (6.75)^{\frac{n-1}{2}} \\ &= 4 \cdot \sqrt{27}^{\,n-1} \\ &< \sqrt{27}^{\,n}. \end{aligned} \tag{1.5}$$

Thus by (1.4) and (1.5) we find the following upper bound for T_2:

$$T_2 < \sqrt{27}^{\,n}. \tag{1.6}$$

In addition, the prime decomposition of $\binom{3n}{2n}$ yields the following upper bound for T_1:

$$T_1 < (3n)^{\pi(\sqrt{3n})}. \tag{1.7}$$

See [3, p. 24]. Then by virtue of Lemma 1.1(2), equality (1.3), and the inequalities(1.6), and (1.7) we find

$$(6.5)^n < T_1 T_2 T_3 < (3n)^{\pi(\sqrt{3n})} \sqrt{27}^{\,n} T_3,$$

which implies that

$$T_3 > \left(\frac{6.5}{\sqrt{27}}\right)^n \frac{1}{(3n)^{\pi(\sqrt{3n})}}.$$

But $\pi(\sqrt{3n}) \leq \frac{\sqrt{3n}}{2}$. Then

$$T_3 > \left(\frac{6.5}{\sqrt{27}}\right)^n \frac{1}{(3n)^{\sqrt{3n}/2}} > 1, \tag{1.8}$$

where the second inequality follows by Lemma 1.1(4). Consequently, the product T_3 of primes between $2n$ and $3n$ is greater than 1 and therefore the existence of such numbers follows. □

Corollary 1.4. *For any positive integer $n \geq 2$ there exists a prime number p satisfying*

$$n < p < \frac{3(n+1)}{2}.$$

Proof. The result is clear for $n = 2$. For even $n > 2$ the result follows by Theorem 1.3. Assume now that $n = 2k + 1$ for a positive integer $k \geq 1$. Then by Theorem 1.3 there is a prime p satisfying

$$2(k+1) < p < 3(k+1) = \frac{3(n+1)}{2},$$

and the result follows. □

References

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