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Development of the Taylor expansion approach for nonlinear integro-differential equations

P. Darania and A. Ebadian

Department of Mathematics, Faculty of Science
Urmia University, P. O. Box 165, Urmia, Iran
p.darania@mail.urmia.ac.ir
a.ebadian@mail.urmia.ac.ir

Abstract

In this paper, the Taylor expansion approach is developed for initial value problems for nonlinear integro-differential equations. This method transformed nonlinear integro-differential equation to a matrix equation which corresponds to a system of nonlinear equations with unknown coefficients. Results of approximate solution to test problems are demonstrated.

Mathematics Subject Classifications: 45J05; 47G20

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1 Introduction

Modeling and analysis of physical phenomena in applied sciences often generates nonlinear mathematical problems. Nonlinearity may be an inner feature of the model, i.e., evolution equations with nonlinear terms, or of the problem, i.e., nonlinear boundary conditions. The interplay between applied sciences and mathematics then leads to the development of initial and/or boundary value problems for nonlinear partial differential or integral or integro-differential equations modeling real physical systems. The theory and application of integral and integro-differential equations is an important subject within applied mathematics. Integral and integro-differential equations are used as mathematical models for many and varied physical situations, and also occur as reformulations of other mathematical problems. Since many physical problems are modeled by integral and integro-differential equations, the numerical solutions of such equations have been highly studied by many authors. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integro-differential equations such as Wavelet-Galerkin method [1], Lagrange interpolation method [2] and Tau method [3] and semi analytical-numerical techniques such as Adomian's decomposition method [4]. However, none of them propose a methodical way to solve these equations. Moreover, previous studies require more effort to achieve the results, they are not accurate and usually they are developed for special types of integro-differential equations.

The problems under consideration are nonlinear integro-differential equations of Fredholm and Volterra types in the forms

$$y'(x) = f(x) + \int_a^b K(x,t,y(t))dt, \qquad a \leq x \leq b,$$

and

$$y'(x) = f(x) + \int_a^x K(x,t,y(t))dt,$$

where a, b are constants, $f(x), K(x,t,y)$ are known functions and $y(x)$ is a solution to be determined.

Actually few theoretical methods in some several cases for this equations are known. Detailed descriptions and analyzes of these methods may be found in [5] and references therein. Here, we will consider the special case of this equations, when the form of nonlinearity is: $K(x,t,y) = K(x,t)g(t,y(t))$. In this equations without loss of generality, we assume that $a = 0$. It will be come clear that following analysis can be readily extended to every $a \in R$. So, by this assumption we will consider the following nonlinear integro-differential equations

$$y'(x) = f(x) + \int_0^b K(x,t)\, g(t,y(t))dt, \qquad 0 \leq x \leq b, \tag{1}$$

and

$$y'(x) = f(x) + \int_0^x K(x,t)\, g(t,y(t))dt, \tag{2}$$

with initial condition $y(0) = y_0$.

The object of the present article is to present developments of the operational approach of the Taylor polynomials for nonlinear integro-differential equations. We proceed to investigate the numerical solvability of equations (1) and (2), by a Taylor expansion approach. Finally, some numerical results are presented in the final section which support the theoretical results obtained in this paper. Throughout this paper, we assume that this equations has a unique solution $y(x)$ to be determined. Existence and uniqueness results for equations (1) and (2), may be found in [5], [6] and references therein.

2 Analysis of the Method

In this section, we give some applications of the proposed scheme for obtaining the approximate solution of the nonlinear integro-differential equations.

2.1 Ferdholm integro-differential equations

Consider the nonlinear integro-differential equation (1) with initial condition $y(0) = y_0$ where $f(x), K(x,t), g(t,y)$ are known functions having $(n+1)$th derivatives on an interval $[0,b]$, b is a constant and the solution is expressed in the form:

$$y(x) = \sum_{n=0}^{N+1} \frac{1}{n!}\, y^{(n)}(0) x^n, \tag{3}$$

which is a Taylor polynomial of degree $N+1$ at $x = 0$, where $y(0) = y_0$ and $y^{(n)}(0), n = 1, \cdots, N+1$ are coefficients to be determined.

We set $g(t, y(t)) = G(y(t))$, where G are known smooth functions with $G(y(t))$ nonlinear in $y(t)$. Then by substituting the Taylor expansion of $G(y(t))$ at $y(t) = 0$, it follows

$$g(t, y(t)) = G(y(t)) = \sum_{l=0}^{N} W(l)\, Y_l(t),$$

where $W(l) = \frac{1}{l!}\frac{d^l}{dy^l}G(y)|_{y=0}$ and $Y_l(t) = y^l(t)$.

So the equation (1) can be written as follows

$$y'(x) = f(x) + \sum_{l=0}^{N} W(l) \int_0^b K(x,t) Y_l(t) dt. \tag{4}$$

To obtain the solution of equation (1), we first differentiate it n times with respect to x:

$$y^{(n+1)}(x) = f^{(n)}(x) + \sum_{l=0}^{N} W(l) \int_0^b \frac{\partial^n K(x,t)}{\partial x^n} Y_l(t) dt. \tag{5}$$

By substituting $x = 0$ in the equation (5), we obtain

$$y^{(n+1)}(0) = f^{(n)}(0) + \sum_{l=0}^{N} W(l) \int_0^b \Big(\frac{\partial^n K(x,t)}{\partial x^n} \Big|_{x=0} Y_l(t)\Big)\, dt. \tag{6}$$

For $l = 0$, we have $Y_0(t) = 1$ and for $l > 0$, we obtain the Taylor expansion of $Y_l(t)$ at $t = 0$ in the following form

$$Y_l(t) = \sum_{m=0}^{N} \frac{Y_l^{(m)}(0)}{m!} t^m. \tag{7}$$

Now, by substitution of equation (7) in the equation (6), we have

$$y^{(n+1)}(0) = f^{(n)}(0) + \sum_{l=0}^{N} W(l) \int_0^b \frac{\partial^n K(x,t)}{\partial x^n}\Bigg|_{x=0} \left[\sum_{m=0}^{N} \frac{Y_l^{(m)}(0)}{m!} t^m\right] dt,$$

or briefly

$$y^{(n+1)}(0) = f^{(n)}(0) + \sum_{l=0}^{N}\sum_{m=0}^{N} K_{nm,\,l}\, Y_l^{(m)}(0), \tag{8}$$

where

$$K_{nm,\,l} = \frac{1}{m!} W(l) \int_0^b \frac{\partial^n K(x,t)}{\partial x^n}\Bigg|_{x=0} t^m dt.$$

$$(n, m, l = 0, 1, \cdots, N)$$

The quantities $Y_l^{(m)}(0)$ for $m = 0, 1, \cdots, N$ in equation (8) can be found from the $Y_0(t) = 1$ and permutation relation

$$Y_l^{(m)}(0) = \begin{cases} \sum_{t_1+t_2+\cdots+t_l=m} \binom{m}{t_1 t_2 \cdots t_l}\; y^{(t_1)}(0) y^{(t_2)}(0) \cdots y^{(t_l)}(0) & l > 0, \quad m = 0, 1, \cdots, N, \\ 0 & l = 0, \quad m \neq 0, \\ 1 & l = m = 0, \end{cases} \tag{9}$$

where $\binom{m}{t_1 t_2 \cdots t_l} = \frac{m!}{t_1! t_2! \cdots t_l!}$, $(t_1, t_2, \cdots, t_l$ are positive integers or zero). Note that the above relation can be obtained from the generalized Leibnitz's rule (dealing with differentiation of product of p-function). For details see e.g. [7,pp.198].

So, the equation (8) is a nonlinear system of $N + 1$ equations for $N + 1$ unknowns $y^{(n+1)}(0)$ $(n = 0, 1, \cdots, N)$, which can be solved numerically by any iterative method. The system (8) can be put in a matrix form as

$$Y - \sum_{l=0}^{N} K_l Y_l^* = F, \tag{10}$$

where Y, F, K_l, Y_l^* are matrices defined by

$$Y = \begin{bmatrix} y^{(1)}(0) \\ y^{(2)}(0) \\ \vdots \\ y^{(N+1)}(0) \end{bmatrix}, \qquad F = \begin{bmatrix} f^{(0)}(0) \\ f^{(1)}(0) \\ \vdots \\ f^{(N)}(0) \end{bmatrix},$$

$$K_l = \begin{bmatrix} K_{00,\,l} & K_{01,\,l} & \cdots & K_{0N,\,l} \\ K_{10,\,l} & K_{11,\,l} & \cdots & K_{1N,\,l} \\ \vdots & \vdots & & \vdots \\ K_{N0,\,l} & K_{N1,\,l} & \cdots & K_{NN,\,l} \end{bmatrix}, \qquad Y_l^* = \begin{bmatrix} Y^{(0)}(0) \\ Y^{(1)}(0) \\ \vdots \\ Y^{(N)}(0) \end{bmatrix}.$$

From this nonlinear system, the unknown Taylor coefficients $y^{(n+1)}(0)(n = 0, 1, \cdots, N)$ are determined and substituted in (3), thus we obtain the approximate solution of the equation (1) in the following form

$$y(x) \cong \sum_{n=0}^{N+1} \frac{1}{n!}\, y^{(n)}(0)\, x^n.$$

2.2 Volterra integro-differential equations

In this subsection we will consider the nonlinear Volterra integro-differential equation (2) with initial condition $y(0) = y_0$, where $f(x), K(x,t), g(t,y)$ are known functions having $(n+1)$th derivatives, and the solution is expressed in the form

$$y(x) = \sum_{n=0}^{N+1} \frac{y^{(n)}(0)}{n!} x^n, \tag{11}$$

which is a Taylor polynomial of degree $N+1$ at $x = 0$, where $y(0) = y_0$ and $y^{(n)}(0), n = 1, 2, \cdots, N+1$ are coefficients to be determined.

According to what we have done in section 2.1, by substituting of the Taylor expansion of g in the equation (2) we have

$$y'(x) = f(x) + \sum_{l=0}^{N} W(l) \int_0^x K(x,t) Y_l(t) dt. \tag{12}$$

Now, by differentiate it n times with respect to x, we have

$$y^{(n+1)}(x) = f^{(n)}(x) + \sum_{l=0}^{N} W(l) V^{(n)}(x), \tag{13}$$

where

$$V^{(n)}(x) = \frac{d^n}{dx^n} \int_0^x K(x,t) Y_l(t) dt.$$

Obviously, for $n = 0$:

$$V(x) = \int_0^x K(x,t) Y_l(t) dt,$$

and for $n > 0$, by applying successively n times the Leibnitz's rule to the integral $V(x)$ we have

$$V^{(n)}(x) = \sum_{j=0}^{n-1} [h_j(x) Y_l(x)]^{(n-j-1)} + \int_0^x \frac{\partial^n K(x,t)}{\partial x^n} Y_l(t)\, dt, \tag{14}$$

where

$$h_j(x) = \left. \frac{\partial^j K(x,t)}{\partial x^j} \right|_{t=x}.$$

From the Leibnitz's rule , we evaluate $[h_j(x)Y_l(x)]^{(n-j-1)}$ as follows

$$[h_j(x)Y_l(x)]^{(n-j-1)} = \sum_{m=0}^{n-j-1} \binom{n-j-1}{m} h_j^{(n-m-j-1)}(x)Y_l^{(m)}(x),$$

and by substituting the above relation in equation (14), we obtain

$$V^{(n)}(x) = \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} \left\{ \binom{n-j-1}{m} h_j^{(n-m-j-1)}(x)Y_l^{(m)}(x) \right\} + \int_0^x \frac{\partial^n K(x,t)}{\partial x^n} Y_l(t)\, dt. \tag{15}$$

Following [8], we have

$$\sum_{j=0}^{n-1} \sum_{m=0}^{n-j-1} (\cdots) = \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} (\cdots).$$

Now, we set $x = c$ in relations (14) and (16), then we obtain

$$\begin{aligned} y^{(n+1)}(c) = f^{(n)}(c) &+ \textstyle\sum_{l=0}^{N} W(l) \left\{ \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} \binom{n-j-1}{m} h_j^{(n-m-j-1)}(c)Y_l^{(m)}(c) \right\} \\ &+ \textstyle\sum_{l=0}^{N} W(l) \left\{ \int_0^c \frac{\partial^n K(x,t)}{\partial x^n} \Big|_{x=c} Y_l(t)\, dt \right\}. \end{aligned}$$

By substituting the Taylor expansion of $Y_l(t)$ at $t = c$ for $l > 0$ in above equation we obtain

$$\begin{aligned} y^{(n+1)}(c) = f^{(n)}(c) &+ \textstyle\sum_{l=0}^{N} W(l) \left\{ \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} \binom{n-j-1}{m} h_j^{(n-m-j-1)}(c)Y_l^{(m)}(c) \right\} \\ &+ \textstyle\sum_{l=0}^{N} W(l) \left\{ \int_0^c \frac{\partial^n K(x,t)}{\partial x^n} \Big|_{x=c} \left[\sum_{m=0}^{N} \frac{1}{m!} Y_l^{(m)}(c)(t-c)^m \right] dt \right\}, \end{aligned}$$

or briefly

$$y^{(n+1)}(c) = f^{(n)}(c) + \sum_{l=0}^{N} \left\{ \sum_{m=0}^{n-1} H_{nm,l} Y_l^{(m)}(c) + \sum_{m=0}^{N} T_{nm,l} Y_l^{(m)}(c) \right\},$$

in other words

$$y^{(n+1)}(c) = f^{(n)}(c) + \sum_{l=0}^{N} \left\{ \sum_{m=0}^{n-1} (H_{nm,l} + T_{nm,l}) Y_l^{(m)}(c) + \sum_{m=n}^{N} T_{nm,l} Y_l^{(m)}(c) \right\}, \tag{16}$$

where for $n = 0$

$$\sum_{m=0}^{n-1} H_{nm,l} + T_{nm,l} Y_l^{(m)}(c) = 0, \qquad (l = 0, 1, \cdots, N)$$

for $n < m$ we have

$$H_{nm,l} = 0, \qquad (l = 0, 1, \cdots, N)$$

and for $n = 1, 2, \cdots, \quad m = 0, 1, \cdots, n-1 \qquad (n > m)$ we have

$$H_{nm,l} = \sum_{j=0}^{n-m-1} W(l) \binom{n-j-1}{m} h_j^{(n-m-j-1)}(c).$$

Also for $n, m, l = 0, 1, 2, \cdots, N$ we have

$$T_{nm,l} = W(l) \frac{1}{m!} \int_0^c \frac{\partial^n K(x,t)}{\partial x^n} \bigg|_{x=c} (t-c)^m \, dt.$$

The quantities $Y_l^{(m)}(c)$ for $m = 0, 1, \cdots, N$ in equation (16) can be found from the permutation relations (9).

If we take $n, m = 0, 1, \cdots, N$ then equation (17) becomes

$$\begin{gathered} y^{(1)}(c) = f^{(0)}(c) + \textstyle\sum_{l=0}^N \sum_{m=0}^N T_{0m,l} Y_l^{(m)}(c), \\ y^{(n+1)}(c) = f^{(n)}(c) + \textstyle\sum_{l=0}^N \left\{ \sum_{m=0}^{n-1} (H_{nm,l} + T_{nm,l}) Y_l^{(m)}(c) + \sum_{m=n}^N T_{nm,l} Y_l^{(m)}(c) \right\}, \end{gathered} \tag{17}$$

which is a nonlinear system of $N+1$ equations for $N+1$ unknowns $y^{(n+1)}(c)(n = 0, 1, \cdots, N)$, which can be solved numerically by any standard methods.

This system can be written as a matrix form

$$Y - \sum_{l=0}^N T_l Y_l^* = F, \tag{18}$$

where Y, F, Y_l^*, T_l are matrices defined by

$$Y = \begin{bmatrix} y^{(1)}(c) \\ y^{(2)}(c) \\ \vdots \\ y^{(N+1)}(c) \end{bmatrix}, \qquad F = \begin{bmatrix} f^{(0)}(c) \\ f^{(1)}(c) \\ \vdots \\ f^{(N)}(c) \end{bmatrix},$$

$$T_l = \begin{bmatrix} T_{00,l} & T_{01,l} & \cdots & T_{0N,l} \\ T_{10,l} + H_{10,l} & T_{11,l} & \cdots & T_{1N,l} \\ \vdots & \vdots & & \vdots \\ T_{N0,l} + H_{N0,l} & T_{N1,l} + H_{N1,l} & \cdots & T_{NN,l} \end{bmatrix}, \qquad Y_l^* = \begin{bmatrix} Y^{(0)}(c) \\ Y^{(1)}(c) \\ \vdots \\ Y^{(N)}(c) \end{bmatrix}.$$

To make easy the calculations, we set $c = 0$, then $T_{nm,l} = 0$ and the system (17) becomes

$$\begin{gathered} y'(0) = f^{(0)}(0) + \textstyle\sum_{l=0}^N \sum_{m=0}^N T_{0m,l} Y_l^{(m)}(0), \\ y^{(n+1)}(0) = f^{(n)}(0) + \textstyle\sum_{l=0}^N \sum_{m=0}^{n-1} H_{nm,l} Y_l^{(m)}(0), \\ n = 1, 2, \cdots; \quad m = 0, 1, \cdots, N. \end{gathered}$$

By this assumption, in the system (18) we have

$$T_l = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ H_{10,l} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ H_{N0,l} & H_{N1,l} & \cdots & 0 \end{bmatrix}.$$

3 Error analysis

In this section, we perform the estimating error for the integro-differential integral equations. Since the truncated Taylor series or the corresponding polynomial expansion is an approximate solution of equations (1) and (2), then by substituting the solutions $y^{(n)}(x)$ $(n = 0, 1, \cdots, N + 1)$, in equation (3), we have

$$e(x) = \left| y(x) - \sum_{n=0}^{N+1} \frac{1}{n!} \, y^{(n)}(0) \, x^n \right|,$$

where $e(x)$ is defined as an error function.

If we set $x = x_r$,

$$e(x_r) = \left| y(x_r) - \sum_{n=0}^{N+1} \frac{1}{n!} \, y^{(n)}(0) \, x_r^n \right|,$$

then our aim is: $e(x_r) \le 10^{-k_r}$ (k_r is any positive integer). If we prescribe , $\max\left(10^{-kr}\right) = 10^{-k}$, then we increase N as far as the following inequality holds at each points x_r

$$e(x_r) \le 10^{-k}.$$

In other words, by increasing N the error function $e(x_r)$ approaches to zero.

4 Numerical example

In this section, we report on numerical results of some test problems solved by the proposed method of this article. We consider the following test problems:

Example 1. Let us consider the nonlinear integro-differential equation with algebraic nonlinearity

$$\begin{cases} y'(x) = 1 - \frac{1}{3}x^3 + \int_0^1 xy^2(t))\,dt, \quad 0 \le x \le 1 \\ y(0) = 0, \end{cases}$$

the exact solution of this problem is $y(x) = x$. First, let us find the coefficients $W(l), (l = 0, 1, 2, 3)$, the derivation values of the function $f(x)$ at $x = 0$, and the matrices $K_{nm,l}(n, m, l = 0, 1, 2, 3)$ for $N = 3$

$$\begin{array}{ll} W(0) = 0, & W(1) = 0, \\ W(2) = 1, & W(3) = 0, \end{array}$$

$$f(0) = 1, \quad f'(0) = \tfrac{-1}{3}, \quad f''(0) = 1, \quad f'''(0) = 0,$$

$$K_0 = K_1 = K_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then for $N = 3$, from the nonlinear system of equation (9) and initial condition, the coefficients $y^{(n)}(0) \ \ (n = 0, 1, 2, 3, 4)$ are find as follows:

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0, \quad y''''(0) = 0.$$

Substituting these coefficients in the Taylor expansion of $y(x)$ at $x = 0$, we obtain the solution

$$y(x) = \sum_{m=0}^{4} \frac{1}{m!}\, y^{(m)}(0)x^m = x,$$

which is an exact solution.

Example 2. Let us now study the following nonlinear integro-differential equation

$$\begin{cases} y'(x) = 1 - \frac{x}{2} + \frac{xe^{-x^2}}{2} + \int_0^x xte^{-y^2(t)}\,dt, \quad 0 \le x \le 1 \\ y(0) = 0, \end{cases}$$

we apply the method in the case $N = 2$. Then we evaluate the quantities $f^{(n)}(0) \ \ (n = 0, 1, 2)$ and matrices $W(l)$ and $T_l \ \ (l = 0, 1, 2)$ as follows

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 0,$$

$$W(0) = 1, \quad W(1) = 0, \quad W(2) = -1,$$

$$T_0 = T_1 = T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We substitute these values in (18), we have $Y = F$. From this system and initial condition, the coefficients $y^{(n)}(0) \quad (n = 0, 1, 2, 3)$ are computed as: $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0$ and thus the solution of the nonlinear Volterra integro-differential equation becomes : $y(x) = x$, which is an exact solution.

Example 3. In this example we will consider the following nonlinear integro-differential equation of initial value type

$$\begin{cases} y'(x) = -1 + \int_0^x y^2(t)\,dt, \\ y(0) = 0, \end{cases} \tag{19}$$

we apply the method in the case $N = 15$. First, we find the following sparse matrices

$$F = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{16\times 1},$$

$$T_0 = T_1 = T_3 = T_4 = \cdots = T_{15} = \mathbf{0},$$

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \cdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{16\times 16}.$$

Then we evaluate the quantities $W(l), \quad l = 0, 1, 2, \cdots, 15$ as follows

$$\begin{array}{cccc} W(0) = 0, & W(1) = 0, & W(2) = 1, & W(3) = 0, \\ W(4) = W(5) = & \cdots & = W(15) = 0, & \end{array}$$

Then, we have :

$$
\begin{aligned}
&y^{(1)}(0) = -1,\\
&y^{(2)}(0) = 0,\\
&y^{(3)}(0) = 0,\\
&y^{(4)}(0) - 2(y^1)^2(0) = 0,\\
&y^{(5)}(0) - 6y^{(1)}(0)y^{(2)}(0) = 0,\\
&y^{(6)}(0) - 8y^{(1)}(0)y^{(3)}(0) - 6(y^{(2)}(0))^2 = 0,\\
&y^{(7)}(0) - 10y^{(1)}(0)y^{(4)}(0) - 20y^{(2)}(0)y^{(3)}(0) = 0.\\
&y^{(8)}(0) - 12y^{(1)}(0)y^{(5)}(0) - 120(y^{(3)}(0))^3(0) - 30y^{(2)}(0)y^{(4)}(0) = 0,\\
&y^{(9)}(0) - 14y^{(1)}(0)y^{(6)}(0) - 70y^{(3)}(0)y^{(4)}(0) - 42y^{(2)}(0)y^{(5)}(0) = 0,\\
&y^{(10)}(0) - 16y^{(1)}(0)y^{(7)}(0) - 112y^{(3)}(0)y^{(5)}(0) - 70(y^{(4)}(0))^2 - 56y^{(2)}(0)y^{(6)}(0) = 0,\\
&y^{(11)}(0) - 18y^{(1)}(0)y^{(8)}(0) - 72y^{(2)}(0)y^{(7)}(0) - 168y^{(3)}(0)y^{(6)}(0) - 252y^{(4)}(0)y^{(5)}(0) = 0,\\
&y^{(12)}(0) - 20y^{(1)}(0)y^{(9)}(0) - 90y^{(2)}(0)y^{(8)}(0) - 240y^{(3)}(0)y^{(7)}(0)\\
&-420y^{(4)}(0)y^{(6)}(0) - 504(y^{(5)}(0))^2 = 0,\\
&y^{(13)}(0) - 22y^{(1)}(0)y^{(10)}(0) - 110y^{(2)}(0)y^{(9)}(0) - 330y^{(3)}(0)y^{(8)}(0)\\
&-660y^{(4)}(0)y^{(7)}(0) - 924y^{(5)}(0)y^{(6)}(0) = 0,\\
&y^{(14)}(0) - 24y^{(1)}(0)y^{(11)}(0) - 122y^{(2)}(0)y^{(10)}(0) - 440y^{(3)}(0)y^{(9)}(0)\\
&-990y^{(4)}(0)y^{(8)}(0) - 1584y^{(5)}(0)y^{(7)}(0) - 924(y^{(6)})^2(0) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&y^{(15)}(0) - 26y^{(1)}(0)y^{(12)}(0) - 156y^{(2)}(0)y^{(11)}(0) - 572y^{(3)}(0)y^{(10)}(0)\\
&-1430y^{(4)}(0)y^{(9)}(0) - 2574y^{(5)}(0)y^{(8)}(0) - 3432y^{(6)}(0)y^{(7)}(0) = 0,\\
&y^{(16)}(0) - 28y^{(1)}(0)y^{(13)}(0) - 182y^{(2)}(0)y^{(12)}(0) - 728y^{(3)}(0)y^{(11)}(0)\\
&-2002y^{(4)}(0)y^{(10)}(0) - 4004y^{(5)}(0)y^{(9)}(0) - 6006y^{(6)}(0)y^{(8)}(0) - 3432(y^{(7)}(0))^2 = 0.
\end{aligned}
$$

Thus, the coefficients are obtained as

$$
\begin{array}{lllll}
y^{(1)}(0)) = -1, & y^{(2)}(0) = 0, & y^{(3)}(0) = 0, & y^{(4)}(0) = 2, & y^{(5)}(0) = 0,\\
y^{(6)}(0)) = 0, & y^{(7)}(0) = -20, & y^{(8)}(0) = 0, & y^{(9)}(0) = 0, & y^{(10)}(0) = 600,\\
y^{(11)}(0)) = 0, & y^{(12)}(0) = 0, & y^{(13)}(0) = -39600, & y^{(14)}(0) = 0, & y^{(15)}(0) = 0,\\
y^{(16)}(0)) = 3511200. & & & &
\end{array}
$$

By substituting these coefficients and initial condition in (11), we get the approximate solution of the equation (19) in the following form

$$y(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{1}{157248}x^{13} + \frac{37}{158505984}x^{16} + O(18). \quad (20)$$

In the study of El-Sayed al. [4], a series solution was obtained for the integro-differential (19) by using the Adomian's decomposition method (ADM), which is the same with the one we presented in equation (20) up to $O(13)$. However, higher ordered terms are slightly different form the ones we obtained. The same equation was also solved by Avudainayagam et al. [1] using Wavelet-Galerkin method (WGM). Numerical results with comparison to both [1,4] are given in Table 1.

5 Conclusion

In this paper, a variation of Taylor polynomial approach has been used for the approximate solution of nonlinear integro-differential equations of Fredholm and Volterra types. This method transformed nonlinear integro-differential equation to a matrix equation which corresponds to a system of nonlinear equations with unknown coefficients. Finally, by using this system, we find the approximate solution of the integro-differential equations.

Table 1. Comparison of numerical results

x	WGM[1]	ADM[4]	Present method
0	0	0	0
0.0312	-0.0312	-0.0311999	-0.03119992103
0.0625	-0.0625	-0.0624987	-0.06249872844
0.0938	-0.0937	-0.0937035	-0.09379354920
0.1250	-0.1250	-0.1249800	-0.1249796568
0.1562	-0.1562	-0.1561500	-0.1561504020
0.1875	-0.1874	-0.1873970	-0.1873970355
0.2188	-0.2186	-0.2186090	-0.2186091064
0.2500	-0.2497	-0.2496750	-0.2496747212
0.2812	-0.2807	-0.2806800	-0.2806795005
0.3125	-0.3117	-0.3117060	-0.3117064248
0.3438	-0.3426	-0.3426380	-0.3426380098
0.3750	-0.3734	-0.3733560	-0.3733561800
0.4062	-0.4040	-0.4039390	-0.4039385135
0.4375	-0.4345	-0.4344590	-0.4344591026
0.4688	-0.4648	-0.4647950	-0.4647946326
0.5000	-04948	-0.4948230	-0.4948225080
0.5312	-0.5247	-0.5246120	-0.5246119111
0.5625	-0.5542	-0.5542270	-0.5542274425
0.5938	-0.5835	-0.5835420	-0.5835419298
0.6250	-0.6124	-0.6124310	-0.6124306816
0.6562	-0.6410	-0.6409540	-0.6409542265
0.6875	-0.6692	-0.6691670	-0.6691672303
0.7188	-0.6969	-0.6969410	-0.6969414639
0.7500	-0.7242	-0.7241530	-0.7241533482
0.7512	-0.7509	-0.7508550	-0.7251900069
0.8125	-0.7771	-0.7770900	-0.7770901036
0.8438	-0.8027	-0.8027340	-0.8027338635
0.8750	-0.8277	-0.8276670	-0.8276674424
0.9062	-0.8520	-0.8519340	-0.8519341734
0.9375	-0.8756	-0.8755690	-0.8755687291
0.9688	-0.8984	-0.8984520	-0.8984522522
1.0000	-0.9205	-0.9204760	-0.9204757028

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