Statistical Dynamics of Optical Solitons
in a Non-Kerr Law Media

Anjan Biswas

Department of Applied Mathematics and Theoretical Physics
Delaware State University
Dover, DE 19901-2277, USA

Swapan Konar

Department of Applied Physics
Birla Institute of Technology
Mesra, Ranchi-835215, India

Essaid Zerrad

Department of Physics and Pre-Engineering
Delaware State University
Dover, DE 19901-2277, USA

Abstract

The statistical dynamics of optical solitons, in a non-Kerr law media, is studied in this paper. The Langevin equations are derived and it is proved that the solitons travel through a fiber with a fixed mean velocity. The non-linearities that are considered here are the power law, parabolic law and the dual-power law types.
1 INTRODUCTION

The dynamics of pulses propagating in optical fibers has been a major area of research given its potential applicability in all optical communication systems. It has been well established [1, 5, 15, 17-20] that this dynamics is described, to first approximation, by the integrable Nonlinear Schrödinger Equation (NLSE). Here the global characteristics of the pulse envelope can be fully determined by the method of Inverse Scattering Transform (IST) and in many instances, the interest is restricted to the single pulse described by the one soliton form of the NLSE. Typically though, distortions of these pulses arise due to perturbations which are either higher order corrections in the model as derived from the original Maxwell’s equations [13], physical mechanisms not considered at first approximation like Raman effects or external perturbations such as the lumped effect due to the addition of bandwidth limited amplifiers in a communication line. Mathematically, these corrections are seen as perturbations of the NLSE and most of them have been studied thoroughly by regular asymptotic, soliton perturbation or Lie transform methods [13, 14, 16].

Besides the deterministic type perturbations one also needs to take into account, from practical considerations, the stochastic type perturbations. These effects can be classified into three basic types [1]:

1. Stochasticity associated with the chaotic nature of the initial pulse due to partial coherence of the laser generated radiation.

2. Stochasticity due to random nonuniformities in the optical fibers like the fluctuations in the values of dielectric constant the random variations of the fiber diameter and more.

3. The chaotic field caused by a dynamic stochasticity might arise from a periodic modulation of the system parameters or when a periodic array
of pulses propagate in a fiber optic resonator.

Thus, stochasticity is inevitable in optical soliton communications [1-4]. Stochasticity are basically of two types namely homogenous and nonhomogenous [11].

1. In the inhomogenous case the stochasticity is present in the input pulse of the fiber. So the parameter dynamics are deterministic but however the initial values are random.

2. In the homogenous case the stochasticity originates due to the random perturbation of the fiber like the density fluctuation of the fiber material or the random variations in the fiber diameter etc.

2 MATHEMATICAL FORMULATION

The dimensionless form of the Nonlinear Schrödinger’s Equation (NLSE) is given by [5]

\[ iq_t + \frac{1}{2} q_{xx} + F(|q|^2) q = 0. \]  

(1)

where \( F \) is a real-valued algebraic function and the smoothness of the complex function \( F(|q|^2) q : C \mapsto C \) is necessary. Considering the complex plane \( C \) as a two-dimensional linear space \( R^2 \), the function \( F(|q|^2) q \) is \( k \) times continuously differentiable so that [5, 9]

\[ F\left(|q|^2\right) q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n) \times (-m,m); R^2\right) \]  

(2)

Equation (1) is a nonlinear partial differential equation (PDE) that is not integrable, in general. The special case, \( F(s) = s \), also known as the Kerr law of nonlinearity, in which case it reduces to the cubic Schrödinger’s equation, is integrable by the method of Inverse Scattering Transform (IST) [1, 13]. The IST is the nonlinear analog of Fourier transform that is used for solving the linear partial differential equations. Schematically, the IST and the technique of Fourier transform are similar [13]. The solutions are known as solitons. The general case \( F(s) \neq s \) takes it away from the IST picture. Equation (1),
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physically, represents the propagation of solitons through an optical fiber.

The three conserved quantities or \textit{integrals of motion} \cite{5, 9, 13} are the energy \((E)\) or \(L_2\) norm, linear momentum \((M)\) and the Hamiltonian \((H)\) that are respectively given by

\[
E = \int_{-\infty}^{\infty} |q|^2 dx
\]  

\[
M = \frac{i}{2} \int_{-\infty}^{\infty} (q^* q_x - q q_x^*) dx
\]  

\[
H = \int_{-\infty}^{\infty} \left[ \frac{1}{2} |q_x|^2 - f(I) \right] dx
\]

where

\[
f(I) = \int_{0}^{I} F(\xi) d\xi
\]

and the intensity \(I\) is given by \(I = |q|^2\). The soliton solution of (1), although not integrable, is assumed to be given in the form \cite{5}

\[
q(x, t) = A(t) g \{B(t) \{x - \bar{x}(t)\}\} e^{i\phi(x,t)}
\]  

where

\[
\frac{d\bar{x}}{dt} = v
\]  

with

\[
\frac{\partial \phi}{\partial x} = -\kappa
\]

and

\[
\frac{\partial \phi}{\partial t} = \frac{B^2}{2} \frac{I_{0,0,2,0}}{I_{0,2,0,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0,0}} \int_{-\infty}^{\infty} g^2(s) F(A^2 g^2(s)) ds
\]

Here, the following integral is defined

\[
I_{l,m,n,p} = \int_{-\infty}^{\infty} \tau^l s^m \left( \frac{d^2 g}{d\tau^2} \right)^n \left( \frac{d s}{d\tau} \right)^p d\tau
\]
for non-negative integers $l, m, n$ and $p$ with $\tau = B(t)(x - \bar{x}(t))$. In (7), the function $g$ represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (1). Also, $\phi(x,t)$ represents the phase of the soliton while the width of the soliton $B(t)$ is related to the amplitude $A(t)$ as $B(t) = \lambda ((A(t))$ where the functional form of $\lambda$ is known if the law of nonlinearity in (1) is given. Finally, $\bar{x}(t)$ gives the mean position of the soliton so that $v$ in (8) represents the velocity of the soliton. For such a general form of the soliton given by (6), the integrals of motion, from (3), (4) and (5), respectively reduce to

$$E = \int_{-\infty}^{\infty} |q|^2 \, dx = \frac{A^2}{B} I_{0,2,0,0}$$  \hspace{1cm} (12)$$

$$M = \frac{i}{2} \int_{-\infty}^{\infty} (qq^* - q^*qx) \, dx = -\kappa \frac{A^2}{B} I_{0,2,0,0}$$  \hspace{1cm} (13)$$

$$H = \int_{-\infty}^{\infty} \left[ \frac{1}{2} |q_x|^2 - f \left( |q|^2 \right) \right] \, dx$$

$$= \frac{A^2 B}{2} I_{0,0,2,0} + \frac{\kappa^2 A^2}{2B} I_{0,2,0,0} - \int_{-\infty}^{\infty} \int_{0}^{I} F(s) ds dx$$  \hspace{1cm} (14)$$

For the soliton assumed in (7), the parameters are now defined as [5]

$$\kappa(t) = \frac{i}{2} \frac{\int_{-\infty}^{\infty} (qq^* - q^*qx) \, dx}{\int_{-\infty}^{\infty} |q|^2 \, dx}$$  \hspace{1cm} (15)$$

$$\bar{x}(t) = \frac{\int_{-\infty}^{\infty} x |q|^2 \, dx}{\int_{-\infty}^{\infty} |q|^2 \, dx}$$  \hspace{1cm} (16)$$

The velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt} = -\kappa = \frac{\partial \phi}{\partial x}$$  \hspace{1cm} (17)$$

From (12), (13) and (15), the parameter dynamics of the unperturbed soliton is as follows

$$\frac{dE}{dt} = 0$$  \hspace{1cm} (18)$$

$$\frac{dM}{dt} = 0$$  \hspace{1cm} (19)$$
Here, (10) is obtained by differentiating (7) with respect to \( t \) and subtracting from its conjugate while using (1). The parameter dynamics for the amplitude and the width of the soliton individually can be obtained for the special cases of \( F(s) \) once the functional form of \( F \), that represents the type of nonlinearity, is known.

\[ \frac{d\kappa}{dt} = 0 \]  

(20)

2.1 PERTURBATION TERMS

The NLSE along with its perturbation terms is given by

\[ iq_t + \frac{1}{2}q_{xx} + F(|q|^2)q = i\epsilon R[q, q^*] \]  

(21)

Here \( R \) is a spatio-differential operator while the perturbation parameter \( \epsilon \), with \( 0 < \epsilon \ll 1 \), represents the relative width of the spectrum in fiber optics that arises due to quasi-monochromaticity [13]. In presence of perturbation terms, as in (21), the integrals of motion are modified. In most instances, a consequence of this is an adiabatic deformation of the soliton parameters like its amplitude, width, frequency and velocity accompanied by small amounts of radiation or small amplitude dispersive waves. The adiabatic parameter dynamics and the evolution of energy, in presence of perturbation terms, neglecting the radiation, are [5-8]

\[ \frac{dE}{dt} = \epsilon \int_{-\infty}^{\infty} (q^* R + q R^*) \, dx \]  

(22)

\[ \frac{dM}{dt} = i\epsilon \int_{-\infty}^{\infty} (q_x^* R - q_x R^*) \, dx \]  

(23)

Equations (22) and (23) lead to

\[ \frac{d\kappa}{dt} = \frac{\epsilon}{I_{0,2,0,0}} \frac{B}{A^2} \left[ i \int_{-\infty}^{\infty} (q_x^* R - q_x R^*) \, dx - \kappa \int_{-\infty}^{\infty} (q^* R + q R^*) \, dx \right] \]  

(24)

Equation (16), for the perturbed NLSE (21) gives

\[ \frac{d\bar{x}}{dt} = -\kappa + \frac{\epsilon}{I_{0,2,0,0}} \frac{B}{A^2} \int_{-\infty}^{\infty} x (q^* R + q R^*) \, dx \]  

(25)
while (10) extends to
\[
\frac{\partial \phi}{\partial t} = \frac{B^2 I_{0,0,2,0}}{2 I_{0,2,0,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0,0}} \int_{-\infty}^{\infty} g^2(s) F \left( A^2 g^2(s) \right) ds + \frac{i\epsilon}{I_{0,2,0,0}} B \frac{1}{2A^2} \int_{-\infty}^{\infty} (qR^* - q^* R) \, dx
\]

(26)

In this paper, the perturbation terms that are going to be considered are given by
\[
R = \delta |q|^{2m} q + \beta q_{xx} - \gamma q_{xxx} + \lambda \left( |q|^2 q \right)_x + \nu \left( |q|^2 \right)_x q + \sigma(x, t)
\]

(27)

For the perturbation terms, in (27), \(\delta < 0\) is the nonlinear damping coefficient \([1, 5, 9]\), \(\beta\) is the bandpass filtering term \([10, 20]\). Also, \(\lambda\) is the self-steepening coefficient for short pulses \([13]\) (typically \(\leq 100\) femto seconds), \(\nu\) is the higher order dispersion coefficient \([13]\) and \(\gamma\) is the coefficient of the third order dispersion \([13]\). The perturbation terms due to \(\alpha\) and \(\beta\) are of non-Hamiltonian or non-conservative type while those due to \(\lambda\), \(\nu\) and \(\gamma\) are of Hamiltonian or conservative type \([5, 13]\).

It is known that the NLSE, as given by (1), does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and so higher order dispersion terms come in. If the group velocity dispersion is close to zero, one needs to consider the third order dispersion for performance enhancement along trans-oceanic distances. Also, for short pulse widths where group velocity dispersion changes, within the spectral bandwidth of the signal cannot be neglected, one needs to take into account the presence of the third order dispersion \([17]\). The perturbation terms due to \(\alpha\) and \(\beta\) are of non-Hamiltonian or non-conservative type while those due to \(\lambda\), \(\nu\) and \(\gamma\) are of Hamiltonian or conservative type \([5, 13]\).

The amplifiers, although needed to restore the soliton energy, introduces noise originating from amplified spontaneous emission (ASE). To study the impact of noise on soliton evolution, the evolution of the mean free velocity of the
soliton due to ASE will be studied in this paper. In case of lumped amplification, solitons are perturbed by ASE in a discrete fashion at the location of the amplifiers. It can be assumed that noise is distributed all along the fiber length since the amplifier spacing satisfies $z_a \ll 1$ [8, 15]. In (27), $\sigma(x,t)$ represents the Markovian stochastic process with Gaussian statistics and is assumed that $\sigma(x,t)$ [6-8, 11] is a function of $t$ only so that $\sigma(x,t) = \sigma(t)$. Now, the complex stochastic term $\sigma(t)$ can be decomposed into real and imaginary parts as

$$\sigma(t) = \sigma_1(t) + i\sigma_2(t)$$

is further assumed to be independently delta correlated in both $\sigma_1(t)$ and $\sigma_2(t)$ with

$$\langle \sigma_1(t) \rangle = \langle \sigma_2(t) \rangle = \langle \sigma_1(t)\sigma_2(t') \rangle = 0$$

$$\langle \sigma_1(t)\sigma_1(t') \rangle = 2D_1\delta(t-t')$$

$$\langle \sigma_2(t)\sigma_2(t') \rangle = 2D_2\delta(t-t')$$

where $D_1$ and $D_2$ are related to the ASE spectral density. In this paper, it is assumed that $D_1 = D_2 = D$. Thus,

$$\langle \sigma(t) \rangle = 0$$

and

$$\langle \sigma(t)\sigma(t') \rangle = 2D\delta(t-t')$$

In soliton units, one gets [15],

$$D = \frac{F_n F_G}{N_{ph} z_a}$$

where $F_n$ is the amplifier noise figure, while

$$F_G = \frac{(G - 1)^2}{G \ln G}$$

is related to the amplifier gain $G$ and finally $N_{ph}$ is the average number of photons in the pulse propagating as a fundamental soliton [15].
Thus, in presence of these perturbation terms, the soliton parameter dynamics changes, by virtue of (22) and (23) to

\[
\frac{dE}{dt} = \frac{2\epsilon A^2}{B} \left[ \delta A^{2m} I_{0,2m+2,0,0} + \beta B \left( B^2 I_{0,1,0,1} - \kappa^2 I_{0,2,0,0} \right) \right] \\
+ 2\epsilon A \int_{-\infty}^{\infty} g(\tau) \left( \sigma_1 \cos \phi + \sigma_2 \sin \phi \right) dx
\] (36)

\[
\frac{dM}{dt} = \frac{2\epsilon}{B} \left[ \kappa A^2 \left( 2\beta AB^3 I_{0,0,3,0} - \delta A^{2m} I_{0,2m+2,0,0} \right) + \beta \kappa A^2 \left( \kappa^2 I_{0,2,0,0} - B^2 I_{0,1,0,1} \right) \right] \\
+ 2\epsilon A \int_{-\infty}^{\infty} \left[ g(\tau) \left( \sigma_1 \cos \phi + \sigma_2 \sin \phi \right) - B \frac{dg}{d\tau} \left( \sigma_2 \cos \phi - \sigma_1 \sin \phi \right) \right] dx
\] (37)

Equations (36) and (37) lead to

\[
\frac{d\kappa}{dt} = \frac{4\epsilon \beta B}{I_{0,2,0,0}} \left( \kappa^2 I_{0,2,0,0} + B^2 I_{0,0,2,0} - B^2 I_{0,1,0,1} \right) \\
- \frac{2\epsilon B}{A I_{0,2,0,0}} \int_{-\infty}^{\infty} \left[ B \frac{dg}{d\tau} \left( \sigma_2 \cos \phi - \sigma_1 \sin \phi \right) + 2\kappa g(\tau) \left( \sigma_1 \cos \phi + \sigma_2 \sin \phi \right) \right] dx
\] (38)

These equations governing the variation of soliton parameters will be individually analysed for Kerr, power, parabolic and dual-power laws of nonlinearity in the following sections.

3 KERR LAW NONLINEARITY

The Kerr law of nonlinearity originates from the fact that a light wave, in an optical fiber, faces nonlinear responses. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distance of propagation that is measured in terms of light wavelength. The origin of nonlinear response is related to the non-harmonic motion of bound electrons under the influence of an applied field. As a result the Fourier amplitude of the induced polarization from the electric dipoles is not linear in the electric field, but involves higher terms in electric field amplitude [13].
For the case of Kerr law nonlinearity, $F(s) = s$ so that $f(s) = s^2/2$. Thus, the NLSE given by (1) modifies to

$$iq_t + \frac{1}{2}q_{xx} + |q|^2q = 0$$

whose 1-soliton solution, as obtained by IST, is given by [5, 9, 13, 15, 20]

$$q(x, t) = \frac{A}{\cosh[B(x - \bar{x}(t))]}e^{i(-\kappa x + \omega t + \sigma_0)}$$

where

$$\omega = \frac{B^2 - \kappa^2}{2}$$

and

$$A = B$$

On comparing (40) with (7)

$$g(\tau) = \frac{1}{\cosh \tau}$$

while the phase is given by

$$\phi(x, t) = -\kappa x + \omega t + \sigma_0$$

where $\omega$ is the wave number and $\sigma_0$ is the center of phase of the soliton.

### 3.1 MATHEMATICAL ANALYSIS

For Kerr law nonlinearity, the NLSE given by (36) has infinitely many integrals of motion. By Noether’s theorem [13], this means that NLSE has an infinite number of symmetries, corresponding to the conserved quantities, which leave the NLSE invariant. The first two integrals of motion are

$$E = \int_{-\infty}^{\infty} |q|^2 dx = 2A$$

and

$$M = i \frac{1}{2} \int_{-\infty}^{\infty} (qq^* - q^*q) dx = -2\kappa A$$
In presence of perturbation terms, the adiabatic variations of the soliton parameters \( A \) and \( \kappa \) that are obtained from (36) and (37) are

\[
\frac{dA}{dt} = \frac{\epsilon}{2} \int_{-\infty}^{\infty} (q^* R + q R^*) dx \tag{47}
\]

and

\[
\frac{d\kappa}{dt} = \frac{\epsilon}{2 A} \left[ i \int_{-\infty}^{\infty} (q^* R - q x R^*) dx - \kappa \int_{-\infty}^{\infty} (q^* R + q R^*) dx \right] \tag{48}
\]

Now substituting the perturbation terms \( R \) from (27) and performing the integrations in (47) and (48) yields

\[
\frac{dA}{dt} = \epsilon \left[ A \left\{ \delta A^{2m} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( m + 1 \right)}{\Gamma \left( m + \frac{3}{2} \right)} - \frac{2}{3} \beta A^2 - 3 \beta \kappa^2 \right\} \right.
\]

\[
+ \pi \left\{ \frac{\sigma_1 \cos(\omega t + \sigma_0) - \sigma_2 \sin(\omega t + \sigma_0)}{\cosh \left( \frac{\pi \kappa}{2A} \right)} \right\} \tag{49}
\]

\[
\frac{d\kappa}{dt} = -\epsilon \left[ \frac{4}{3} \beta A^2 \kappa + \frac{\pi}{2 \eta} \left( 2A^2 + 2 \kappa^2 - \kappa A \right) \left\{ \frac{\sigma_1 \cos(\omega t + \sigma_0) - \sigma_2 \sin(\omega t + \sigma_0)}{\cosh \left( \frac{\pi \kappa}{2A} \right)} \right\} \right] \tag{50}
\]

Equations (47) and (48), as it appears, are difficult to analyse. If the terms with \( \sigma_1 \) and \( \sigma_2 \) are suppressed, the resulting dynamical system has a stable fixed point, namely a sink, given by \((\bar{A}, \bar{\kappa}) = (\bar{A}, 0)\) where

\[
\bar{A} = \left[ \frac{2 \sigma \Gamma \left( m + \frac{3}{2} \right)}{3 \delta \Gamma \left( \frac{1}{2} \right) \Gamma (m + 1)} \right]^{\frac{1}{m + 1}} \tag{51}
\]

Now, linearizing the dynamical system about this fixed point and simplifying gives

\[
\frac{dA}{dt} = -\epsilon \left( A^{2m+1} - \frac{\zeta}{A} \right) \tag{52}
\]

\[
\frac{d\kappa}{dt} = -\epsilon \left[ \kappa - \zeta (1 + A - \kappa) \right] \tag{53}
\]

where

\[
\zeta = \pi \left\{ \frac{\sigma_1 \cos(\omega t + \sigma_0) - \sigma_2 \sin(\omega t + \sigma_0)}{\cosh \left( \frac{\pi \kappa}{2A} \right)} \right\} \tag{54}
\]
Equations (52) and (53) are called the Langevin equations which will now be analyzed to compute the soliton mean drift velocity of the soliton. If the soliton parameters are chosen such that $\zeta A$ is small, then (53) yields

$$\frac{dk}{dt} = -\epsilon [k - \zeta (1 - k)] \quad (55)$$

One can solve (55) for $k$ and eventually the mean drift velocity of the soliton can be obtained. The stochastic phase factor of the soliton is defined by

$$\psi(t, y) = \int_y^t \zeta(s)ds \quad (56)$$

where $t > y$. Assuming that $\sigma$ is a Gaussian stochastic variable one arrives at

$$\langle e^{\psi(t, y)} \rangle = e^{D(t-y)} \quad (57)$$
$$\langle e^{[\psi(t, y) + \psi(t', y')]\rangle} = e^{D\theta} \quad (58)$$

where

$$\theta = 2(t + t' - y - y') - |t - t'| - |y - y'| \quad (59)$$

and

$$\langle \zeta(y)e^{-\psi(t, y)} \rangle = \frac{\partial}{\partial y} \langle e^{-\psi(t, y)} \rangle = De^{D(t-y)} \quad (60)$$

$$\langle \zeta(y)\zeta(y')e^{-[\psi(t, y) - \psi(t', y')]\rangle} = 2D\delta(y - y')e^{D\theta} + \frac{\partial^2}{\partial y \partial y'}e^{D\theta} \quad (61)$$

Now solving (55) with the initial condition as $k(0) = 0$ and using equations (56)-(61) the soliton mean drift velocity is given by

$$\langle k(t) \rangle = -\frac{D}{1 - D} \left\{ 1 - e^{-\epsilon(1-D)t} \right\} \quad (62)$$

From (62), it follows that

$$\lim_{t \to \infty} \langle k(t) \rangle = -\frac{D}{1 - D} \quad (63)$$

so that in the limiting case, for $D < 1$, the mean free velocity of the soliton is given by

$$\langle v \rangle = \frac{D}{1 - D} \quad (64)$$

Thus, for large $t$, $\langle v(t) \rangle$ is constant. If $D > 1$, $\langle k(t) \rangle$ becomes unbounded for large $t$. Similarly, the two point correlation computes out to be

$$\langle k(t)k(t') \rangle = -\frac{D}{1 - 2D} \left\{ e^{[-(1-D)t-t']} - e^{[-(1-2D)t+t'-D|-t'-t|]} \right\} + O(D^2) \quad (65)$$
4 POWER LAW NONLINEARITY

Power law nonlinearity is exhibited in various materials including semiconductors. This law also occurs in media for which higher order photon processes dominate at different intensities. This law is also treated as a generalization to the Kerr law nonlinearity.

For the case of power law nonlinearity, $F(s) = s^p$ so that $f(s) = s^{p+1}/(p+1)$ so that the NLSE, given by (1), modifies to

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = 0$$

(66)

In (66), it is necessary to have $0 < p < 2$ to prevent wave collapse [5, 9] and, in particular, $p \neq 2$ to avoid self-focussing singularity [5]. The soliton solution of (66) is given by [5, 9]

$$q(x, t) = \frac{A}{\cosh^{\frac{1}{p}} [B(x - \bar{x}(t))]} e^{i(\kappa x + \omega t + \sigma_0)}$$

(67)

where

$$\omega = \frac{B^2}{2p^2} - \frac{\kappa^2}{2}$$

(68)

and

$$B = A^p \left( \frac{2p^2}{1 + p} \right)^{\frac{1}{2}}$$

(69)

On comparing (67) with (7), gives

$$g(\tau) = \frac{1}{\cosh^{\frac{1}{p}} \tau}$$

(70)

4.1 MATHEMATICAL ANALYSIS

The first two integrals of motion of the power law nonlinearity are

$$E = \int_{-\infty}^{\infty} |q|^2 dx$$

$$= A^{2-p} \left( \frac{1 + p}{2p^2} \right)^{\frac{1}{2}} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)} = B^{\frac{2-p}{p}} \left( \frac{1 + p}{2p^2} \right)^{\frac{1}{2}} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}$$

(71)
Using these integrals of motion, one can obtain

\[ M = \frac{i}{2} \int_{-\infty}^{\infty} (q^* q_x - q_x^*) dx \]

\[ = 2\kappa A^{2-p} \left( \frac{1 + p}{2p^2} \right)^{\frac{1}{p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)} \]

\[ = 2\kappa B^{2-p} \left( \frac{1 + p}{2p^2} \right)^{\frac{1}{p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)} \]

(72)

Using these integrals of motion, one can obtain

\[ \frac{dA}{dt} = \epsilon \frac{A^{p-1}}{2 - p} \left( \frac{2p^2}{1 + p} \right)^{\frac{4 - p}{2p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \int_{-\infty}^{\infty} (q^* R + qR^*) dx \]

(73)

\[ \frac{dk}{dt} = \epsilon B^{\frac{p-2}{p}} \left( \frac{2p^2}{1 + p} \right)^{\frac{1}{p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \left[ i \int_{-\infty}^{\infty} (q^* x - q_x R^*) dx - \kappa \int_{-\infty}^{\infty} (q^* R + qR^*) dx \right] \]

(74)

Now substituting the perturbation terms \( R \) from (27) and carrying out the integrations in (73) and (74) yields

\[ \frac{dA}{dt} = \frac{2\epsilon \delta}{2 - p} A^{2m+1} \left( \frac{1 + p}{2p^2} \right)^{\frac{4 - p}{2p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \frac{\Gamma \left( \frac{m+1}{p} \right)}{\Gamma \left( \frac{m+1}{p} + \frac{1}{2} \right)} \]

\[ + 2\epsilon \beta \frac{A^{p-1}}{2 - p} \left( \frac{2p^2}{p + 1} \right)^{\frac{4 - p}{2p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \left[ \frac{A^2 B^2}{p^2} \frac{\Gamma \left( \frac{p+1}{p} \right)}{\Gamma \left( \frac{p+1}{p} + \frac{1}{2} \right)} - \frac{A^2}{p^2} \left( \kappa^2 p^2 + B^2 \right) \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)} \right] \]

\[ + 2\epsilon A \frac{2p^2}{1 + p} \left( \frac{4 - p}{2p} \right)^{\frac{1}{p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \left[ \cos \phi \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 \tau} + \sin \phi \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 \tau} \right] \]

(75)

\[ \frac{dk}{dt} = \frac{4\epsilon \beta}{p^2 - \kappa A^2 B^{\frac{p-2}{p}}} \left( \frac{2p^2}{p + 1} \right)^{\frac{1}{p}} \left( \frac{p - 2}{p + 2} \right) \]

\[ + 4\epsilon \kappa A B^{\frac{p-2}{p}} \left( \frac{2p^2}{1 + p} \right)^{\frac{1}{p}} \frac{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{p} \right)} \left[ \frac{B \tanh \tau}{p \cosh^2 \tau} \frac{\kappa^2 \cos \phi - \sigma_1 \sin \phi \right] \]

\[ + \frac{2\kappa}{\cosh^2 \tau} \left( \sigma_1 \cos \phi + \sigma_2 \sin \phi \right) \]

(76)

Equations (75) and (76) are difficult to analyse. If the terms with \( \sigma_1 \) and \( \sigma_2 \) are suppressed, the resulting dynamical system has a stable fixed point, namely a
sink, given by \((\bar{A}, \bar{\kappa}) = (\bar{A}, 0)\) where

\[
\bar{A} = \left[ \frac{2\sigma}{\delta(p+1)} \frac{\Gamma \left( \frac{m+1}{p} + \frac{1}{2} \right)}{\Gamma \left( \frac{m+1}{p} \right)} \left\{ \frac{\Gamma \left( \frac{p+1}{p} \right)}{\Gamma \left( \frac{p+1}{p} + \frac{1}{2} \right)} - \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)} \right\} \right]^{\frac{1}{2(m-p)}} \tag{77}
\]

Now, linearizing the dynamical system about this fixed point gives, after simplification

\[
\frac{dA}{dt} = -\epsilon \left( A^{2m+1} - \frac{\zeta_1^{(1)}}{A} \right) \tag{78}
\]

\[
\frac{d\kappa}{dt} = -\epsilon \left[ \kappa - \zeta_2^{(1)} (1 + A - \kappa) \right] \tag{79}
\]

where

\[
\zeta_1^{(1)} = \sigma_1 \int_{-\infty}^{\infty} \frac{\cos \phi}{\cosh \frac{1}{p} \tau} dx + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin \phi}{\cosh \frac{1}{p} \tau} dx \tag{80}
\]

and

\[
\zeta_2^{(1)} = \int_{-\infty}^{\infty} \left[ B \frac{\tanh \tau}{p \cosh \frac{1}{p} \tau} (\sigma_2 \cos \phi - \sigma_1 \sin \phi) + \frac{2\kappa}{\cosh \frac{1}{p} \tau} (\sigma_1 \cos \phi + \sigma_2 \sin \phi) \right] dx \tag{81}
\]

Similarly, as in the case of Kerr law nonlinearity, these Langevin equations lead to the mean drift velocity of the soliton as

\[
\langle v \rangle = \frac{D}{1 - D} \tag{82}
\]

## 5 PARABOLIC LAW NONLINEARITY

There was little attention paid to the propagation of optical beams in the fifth order nonlinear media, since no analytic solutions were known and it seemed that chances of finding any material with significant fifth order term was slim. However, recent developments have rekindled interest in this area. The optical susceptibility of CdS\(_x\)Se\(_{1-x}\)-doped glasses was experimentally shown to have a considerable \(\chi^{(5)}\), the fifth order susceptibility. It was also demonstrated that there exists a significant \(\chi^{(5)}\) nonlinearity effect in a transparent glass in intense femtosecond pulses at 620 nm [5].
It is necessary to consider nonlinearities higher than the third order to obtain some knowledge of the diameter of the self-trapping beam. It was recognized in 1960s and 70s that saturation of the nonlinear refractive index plays a fundamental role in the self-trapping phenomenon. Higher order nonlinearities arise by retaining the higher order terms in the nonlinear polarization tensor.

For this law, \( F(s) = s + \nu s^2 \) where \( \nu \) is a constant, so that \( f(s) = s^2/2 + \nu s^3/3 \). The form of the GNLSE here is

\[
i q_t + \frac{1}{2} q_{xx} + \left( |q|^2 + \nu |q|^4 \right) q = 0
\]  

(83)

The solution of (83) is now written as [5]

\[
q(x, t) = \frac{A}{[1 + a \cosh \{B (x - \bar{x}(t))\}]^2} e^{i(-\kappa x + \omega t + \sigma_0)}
\]  

(84)

where

\[
\omega = \frac{A^2}{4} - \frac{\kappa^2}{2}
\]  

(85)

and

\[
B = \sqrt{2}A
\]  

(86)

with

\[
a = \sqrt{1 + \frac{4}{3} \nu A^2}
\]  

(87)

On comparing (84) with (7), gives

\[
g(\tau) = \frac{1}{(1 + a \cosh \tau)^2}
\]  

(88)

5.1 MATHEMATICAL ANALYSIS

The first two integrals of motion of the parabolic law nonlinearity are [5]

\[
E = \int_{-\infty}^{\infty} |q|^2 dx = \begin{cases} \sqrt{\frac{3}{2\nu}} \tan^{-1} \left[ \frac{2A \sqrt{\nu}}{3} \right] & : 0 < \nu < \infty \\ \sqrt{-\frac{3}{2\nu}} \tanh^{-1} \left[ \frac{2A \sqrt{-\nu}}{3} \right] & : -\frac{3}{4A^2} < \nu < 0 \end{cases}
\]  

(89)
\[
M = \frac{i}{2} \int_{-\infty}^{\infty} (q q_x^* - q^* q_x) \, dx
\]
\[
= \left\{ \begin{array}{ll}
-\frac{\pi}{2} \sqrt{\frac{2}{\nu}} \tan^{-1} \left[ 2A \sqrt{\frac{2}{3}} \right] & : 0 < \nu < \infty \\
-\frac{\pi}{2} \sqrt{-\frac{3}{2\nu}} \tanh^{-1} \left[ 2A \sqrt{-\frac{2}{3}} \right] & : -\frac{3}{4A^2} < \nu < 0
\end{array} \right. \tag{90}
\]

Using these integrals of motion, one obtains
\[
\frac{dA}{dt} = \epsilon a^2 \sqrt{\frac{2}{\nu}} \int_{-\infty}^{\infty} (q^* R + q R^*) \, dx \tag{91}
\]
\[
\frac{d\kappa}{dt} = \epsilon \frac{B}{A^2} \left[ i \int_{-\infty}^{\infty} (q_x^* R - q_x R^*) \, dx - \kappa \int_{-\infty}^{\infty} (q^* R + q R^*) \, dx \right] \tag{92}
\]

where \( E \) is the energy of the soliton. Now, substituting the perturbation terms \( R \) from (27) and carrying out the integrations in (91) and (92) yields
\[
\frac{dA}{dt} = \frac{\epsilon \delta A^{2m+1}}{2^m a^{m+1}} F \left( m + 1, m + 1, m + 3 \frac{a - 1}{2a} ; \frac{1}{2} \right) B \left( m + 1, \frac{1}{2} \right) \\
+ \frac{\epsilon \sqrt{2} \beta A}{a} \left[ \kappa^2 F \left( 1, 1, \frac{3a}{2}; \frac{a - 1}{2a} \right) B \left( 1, \frac{1}{2} \right) - 2A^2 F \left( 3, 1, \frac{5a}{2}; \frac{a - 1}{2a} \right) B \left( 1, \frac{3}{2} \right) \right] \\
+ \epsilon \frac{B}{A^2} \left[ \sqrt{\frac{2}{\nu}} \int_{-\infty}^{\infty} \frac{\cos \phi}{\cosh \tau} \, dx + \sqrt{\frac{2}{\nu}} \int_{-\infty}^{\infty} \frac{\sin \phi}{\cosh \tau} \, dx \right] \tag{93}
\]
\[
\frac{d\kappa}{dt} = -\frac{\epsilon \beta \kappa A^2}{2} \frac{F \left( 3, 2, \frac{a-1}{2a} ; \frac{1}{2} \right) B \left( 2, 1 \right)}{F \left( 1, 1, \frac{3a}{2}; \frac{a - 1}{2a} \right) B \left( 1, \frac{1}{2} \right)} \\
- \frac{\epsilon \sqrt{2}}{AE} \int_{-\infty}^{\infty} \left[ \frac{a B (\sigma_2 \cos \phi - \sigma_1 \sin \phi) \sinh \tau}{2 \left( 1 + a \cosh \tau \right)^{3/2}} \right. \\
\left. - \frac{2 \kappa (\sigma_1 \cos \phi + \sigma_2 \sin \phi)}{\left( 1 + a \cosh \tau \right)^{3/2}} \right] \, dx \tag{94}
\]

where in (93) and (94), \( F(a, b; c; z) \) is the Gauss' hypergeometric function defined as
\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n) z^n}{\Gamma(c + n) n!} \tag{95}
\]
and \( B(l, m) \) is the beta function that is defined as
\[
B(l, m) = \int_{0}^{1} x^{l-1} (1 - x)^{m-1} \, dx \tag{96}
\]
Now, linearizing the dynamical system about this fixed point gives, after simplification

\[
\frac{dA}{dt} = -\epsilon \left( A^{2m+1} - \zeta_1^{(2)} \right)
\]

(97)

\[
\frac{d\kappa}{dt} = -\epsilon \left[ \kappa - \zeta_2^{(2)} (1 + A - \kappa) \right]
\]

(98)

where \( \bar{A} \) is the fixed point of the amplitude while

\[
\zeta_1^{(2)} = \sigma_1 \int_{-\infty}^{\infty} \frac{\cos \phi}{(1 + a \cosh \tau)^{\frac{1}{2}}} d\tau + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin \phi}{(1 + a \cosh \tau)^{\frac{1}{2}}} d\tau
\]

(99)

and

\[
\zeta_2^{(2)} = \int_{-\infty}^{\infty} \left[ \frac{aB}{2} \frac{\sigma_2 \cos \phi - \sigma_1 \sin \phi}{(1 + a \cosh \tau)^{\frac{3}{2}}} \sinh \tau - \frac{2\kappa}{(1 + a \cosh \tau)^{\frac{1}{2}}} \right] d\tau
\]

(100)

Again, as in the case of Kerr law nonlinearity, these Langevin equations lead to the mean drift velocity of the parabolic law soliton as

\[
\langle v \rangle = \frac{D}{1 - D}
\]

(101)

6 DUAL-POWER LAW NONLINEARITY

An important aspect that has not been addressed with proper perspective is the fact that due to its nonsaturable nature, Kerr nonlinearity is inadequate to describe the soliton dynamics in the ultrahigh bit rate transmission. For example, when transmission bit rate is very high, for soliton formation the peak power of the incident field accordingly become very large. On the other hand higher order nonlinearities may become significant even at moderate intensities in certain materials such as semiconductor doped glass fibers. This problem can be addressed by incorporating the dual-power law nonlinearity in the NLSE.
For this law, \( F(s) = s^p + \nu s^{2p} \) where \( \nu \) is a constant, so that \( f(s) = s^{p+1}/(p+1) + \nu s^{2p+1}/(2p+1) \). The form of the GNLSE here is

\[
iq_t + \frac{1}{2}q_{xx} + \left( |q|^{2p} + \nu |q|^{4p} \right) q = 0 \tag{102}
\]

Equation (102) supports solitary waves of the form [5]

\[
q(x,t) = \frac{A}{\left[ 1 + b \cosh \left\{ B (x - \bar{x}(t)) \right\} \right]^{\frac{1}{2p}}} e^{i(-\kappa x + \omega t + \sigma_0)} \tag{103}
\]

where

\[
\omega = \frac{A^{2p}}{2p+2} - \frac{\kappa^2}{2} \tag{104}
\]

while

\[
B = A^p \left( \frac{2p^2}{1+p} \right)^{\frac{1}{2p}} \tag{105}
\]

with

\[
b = \sqrt{1 + \frac{\nu B^2 (1+p)^2}{2p^2 (1+2p)}} \tag{106}
\]

The dual-power law case the solitons exist for

\[
-\frac{2p^2}{B^2 (1+p)^2} < \nu < 0 \tag{107}
\]

On comparing (103) with (7), gives

\[
g(\tau) = \frac{1}{(1 + b \cosh \tau)^{\frac{1}{2p}}} \tag{108}
\]

### 6.1 MATHEMATICAL ANALYSIS

The first two integrals of motion of the dual-power law nonlinearity are

\[
E = \int_{-\infty}^{\infty} |q|^2 dx = \frac{2A^2}{B^{2p}b^{2p}} F \left( \frac{1}{p}; \frac{1}{2}; \frac{1}{p} + \frac{b-1}{2b} \right) B \left( \frac{1}{p}; \frac{1}{2} \right) \tag{109}
\]

\[
M = \frac{i}{2} \int_{-\infty}^{\infty} (qq_x^* - q^* q_x) dx = \frac{2\kappa A^2}{B^{2p}b^{2p}} F \left( \frac{1}{p}; \frac{1}{2}; \frac{1}{p} + \frac{b-1}{2b} \right) B \left( \frac{1}{p}; \frac{1}{2} \right) \tag{110}
\]
Using these integrals of motion, it is possible to obtain

\[
\frac{dA}{dt} = \epsilon \frac{1}{pLA^{p-1}} \left( \frac{p + 1}{2p^2} \right)^{\frac{1}{2p}} \int_{-\infty}^{\infty} (q^* R + qR^*) dx
\] (111)

\[
\frac{d\kappa}{dt} = \frac{\epsilon}{E} \left[ i \int_{-\infty}^{\infty} (q^*_x R - q_x R^*) dx - \kappa \int_{-\infty}^{\infty} (q^* R + qR^*) dx \right]
\] (112)

where \( E \) is the energy, while

\[
L = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2} \right)} \left[ \frac{(b - 1)(2p + 1)}{2
\nu(1 + p)} \right]^{\frac{1}{p}}
\]

\[
\begin{align*}
&\frac{2\nu^2}{bp^2} \frac{(p + 1)^3}{(b - 1)(2p + 1)^2} \left( \frac{1}{2}; \frac{1}{p} + \frac{1}{p}; \frac{1 - b}{1 + b} \right) \\
&- \frac{2}{B^2} \left( \frac{1}{2}; \frac{1}{p} + \frac{1}{p}; \frac{1 - b}{1 + b} \right) \\
&- \frac{2\nu}{bp^2} \frac{(p + 1)^2}{(b - 1)(2p + 1)^2} \left( \frac{1}{2}; \frac{1}{p} + \frac{1}{p}; \frac{1 - b}{1 + b} \right)
\end{align*}
\] (113)

Now substituting the perturbation terms \( R \) from (27) and carrying out the integrations in (111) and (112) yields

\[
\frac{dA}{dt} = \frac{4\epsilon \delta A^{2m+2}}{Bb \beta A^{m+1} 2^{m+1} p} F \left( \frac{m+1}{p}, \frac{m+1}{p}, \frac{m+1}{p} + 1; \frac{b-1}{2b} \right) B \left( \frac{m+1}{p}, \frac{1}{2} \right) \\
- \frac{4\epsilon \beta A^2}{Bb \beta^2 2^p} \left[ B^2 F \left( 2 + \frac{1}{p}, \frac{1}{p} + \frac{3}{2}; \frac{b-1}{2b} \right) B \left( \frac{1}{p}, \frac{3}{2} \right) \\
+ \frac{\kappa^2}{p} F \left( \frac{1}{p}, \frac{1}{2}; \frac{b-1}{2b} \right) B \left( \frac{1}{p}, \frac{1}{2} \right) \right] \\
+ \frac{2\epsilon}{pLA^{p-2}} \left( \frac{p + 1}{2p^2} \right)^{\frac{1}{2p}} \left[ \sigma_1 \int_{-\infty}^{\infty} \frac{\cos \phi}{(1 + a \cosh \tau)^{\frac{1}{2p}}} dx + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin \phi}{(1 + a \cosh \tau)^{\frac{1}{2p}}} dx \right]
\] (114)

\[
\frac{d\kappa}{dt} = -\frac{\epsilon \beta B^2}{4p^2 A^2} F \left( \frac{2 + \frac{1}{p}, \frac{1}{p} + \frac{1}{p}, \frac{1}{p} + \frac{b-1}{2b}}{2} \right) B \left( \frac{1 + \frac{1}{p}, \frac{1}{2} \right) \\
- \frac{\epsilon}{E} \int_{-\infty}^{\infty} \left[ \frac{aB}{2p} \left( \sigma_2 \cos \phi - \sigma_1 \sin \phi \right) \sinh \tau - \frac{2\kappa}{2p} \left( \sigma_1 \cos \phi + \sigma_2 \sin \phi \right) \right] dx
\] (115)
Now, linearizing the dynamical system about this fixed point gives, after simplification

$$\frac{dA}{dt} = -\epsilon \left( A^{2m+1} - \frac{\zeta^{(3)}_1}{A} \right)$$  \hspace{1cm} (116)$$

$$\frac{d\kappa}{dt} = -\epsilon \left[ \kappa - \zeta^{(3)}_2 (1 + A - \kappa) \right]$$  \hspace{1cm} (117)$$

where $\bar{A}$ is the fixed point of the amplitude while

$$\zeta^{(3)}_1 = \sigma_1 \int_{-\infty}^{\infty} \frac{\cos \phi}{(1 + a \cosh \tau)^{\frac{1}{2p}}} dx + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin \phi}{(1 + a \cosh \tau)^{\frac{1}{2p}}} dx$$  \hspace{1cm} (118)$$

and

$$\zeta^{(3)}_2 = \int_{-\infty}^{\infty} \left[ \frac{aB (\sigma_2 \cos \phi - \sigma_1 \sin \phi) \sinh \tau}{2p} - \frac{2\kappa (\sigma_1 \cos \phi + \sigma_2 \sin \phi)}{(1 + a \cosh \tau)^{\frac{1}{2p}}} \right] dx$$  \hspace{1cm} (119)$$

Once again, as in the case of Kerr law nonlinearity, one can derive the mean drift velocity of the soliton as

$$\langle v \rangle = \frac{D}{1 - D}$$  \hspace{1cm} (120)$$

7 CONCLUSIONS

In this paper, the dynamics of optical solitons with non-Kerr law nonlinearities in presence of perturbation terms, both deterministic as well as stochastic, are studied. The Langevin equations were derived and the corresponding parameter dynamics was studied. The mean drift velocity of the soliton was obtained.

In this study, it was assumed that the stochastic perturbation term $\sigma$ is a function of $t$ only, for simplicity. However, in reality, $\sigma$ is a function of both $x$ and $t$ and thus making it a far more difficult system to analyze although such kind of situations are being presently studied. Although, in this paper the stochastic perturbation due to other non-Kerr law nonlinearities was not studied, for example saturable law, triple power law of nonlinearity amongst many others, the results of those studies are awaited at this time.
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