Int. J. Contemp. Math. Sciences, Vol. 1, 2006, no. 13, 639 - 650

The Method of Lines Solution of the Korteweg-de Vries Equation for Small Times

A. Özdeş and E. N. Aksan*

Department of Mathematics
Faculty of Arts and Science
İnönü University, 44069, Malatya, Turkey

Abstract. In this paper, the application of the method of lines (MOL) to the KdV equation was presented. The MOL approach of the KdV equation led to a system of ordinary differential equations. Solution of the system was obtained by applying Euler's method. In order to show the accuracy of the presented method, two test problem whose exact solutions are known were used. The numerical solutions obtained were compared with not only exact solutions but also other numerical solutions obtained by using various numerical techniques. It was seen that they were in good agreement with each other.

Keywords: KdV equation, The method of lines.

Mathematics Subject Classifications: 35Q53, 65N20

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*Corresponding author. E-mail naksan@inonu.edu.tr

1. Introduction

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation of third order as

$$U_t + \varepsilon U U_x + \mu U_{xxx} = 0 \qquad a \le x \le b \tag{1}$$

where ε and μ are positive parameters. The KdV equation first appeared in an article written by Dutch mathematicians Korteweg and De Vries [5]. They formulated The KdV equation to describe long wave propagation on shallow water. The KdV equation is one of the simplest nonlinear model equations for solitary waves. In the study on the KdV equation, Zabuskay and Kruskal indicated that the wave solutions persisted after interactions and the wave solutions were called as 'solitons' [12]. The soliton concept is the very important in the study of nonlinear wave phenomena. Physically, when two solitons of different amplitudes (and hence, of different speeds) are placed far apart on the real line, the taller (faster) wave is on the left of the shorter (slower) wave, the taller one eventually catches up to the shorter one and then overtakes it. When this happens they undergo a nonlinear interaction according to the KdV equation and emerge from the interaction completely preserved in form and speed with only a phase shift. Thus these two remarkable features: (i) steady progressive pulselike solitons and (ii) the preservation of their shapes and speeds, confirmed the particlelike property of the waves [9]. Many exact solutions of the equation with appropriate initial condition by using scattering theory were given by Gardner et al. [3]. On the other hand, the numerical solutions of the equation with initial and boundary conditions were obtained by the various methods based on finite element methods, finite difference methods and spectral methods [2,7,8]. Recently, the papers including numerical solutions for the KdV equation at the small time by using various numerical techniques were published [1,4,6].

In this study, the method of lines (MOL) solution of the equation for the small times was presented. The MOL consists of converting the partial differential equation with auxiliary conditions into a system of ordinary differential equations with corresponding auxiliary conditions. The performance of the method was tested on known two model problems.

2. The Method of Lines Solution of the KdV equation

We consider KdV equation (1) with initial condition

$$U(x,0) = g(x) \qquad a \le x \le b \tag{2}$$

and the boundary conditions

$$U(a,t) = 0 \quad , \quad U(b,t) = 0 \qquad t > 0 \tag{3}$$

where $g(x)$ is the prescribed function. The solution domain of the KdV equation (1) is the rectangle $a \le x \le b$, $0 \le t \le T$. Let us subdivide it into uniform rectangular meshes by the lines $x_i = ih$ $(i = 0,1,2,...,N)$, where $h = (b-a)/N$ and the lines $t_j = jk$ $(j = 0,1,2,...)$, where $jk = T$. We replace the partial derivatives depend on spatial variables, $\partial U / \partial x$ and $\partial^3 U / \partial x^3$, in KdV equation (1) with known finite difference approximations at point x_i [10]. This yields a system of ordinary differential equations depend on t in the form

$$\frac{dU}{dt} = f(U_i) \qquad i = 1,2,...,N-1 \tag{4}$$

where

$$f(U_i) = \begin{cases} -\mu \dfrac{-5U_i + 18U_{i+1} - 24U_{i+2} + 14U_{i+3} - 3U_{i+4}}{2h^3} - \varepsilon U_i \dfrac{-3U_i + 4U_{i+1} - U_{i+2}}{2h} & ; \quad i = 1,2 \\ -\mu \dfrac{U_{i-3} - 8U_{i-2} + 13U_{i-1} - 13U_{i+1} + 8U_{i+2} - U_{i+3}}{8h^3} - \varepsilon U_i \dfrac{-U_{i-1} + U_{i+1}}{2h} & ; \quad i = 3,4,\ldots,N-1 \\ -\mu \dfrac{3U_{i-4} - 14U_{i-3} + 24U_{i-2} - 18U_{i+1} + 5U_i}{2h^3} - \varepsilon U_i \dfrac{U_{i-2} - 4U_{i-1} + 3U_i}{2h} & ; \quad i = N-2, N-1 \end{cases}$$

Thus, we have the system of differential equations of one independent variable t. This system can be easily solved by using Euler's method [13].

If we apply Euler's method to the Eq. (4), we obtain

$$U^{j+1} = U^j + k\, f\left(U^j\right) \quad j = 0,1,2,\ldots$$

Clearly this system can be expressed as

$$U_i^{j+1} = \begin{cases} U_i^j - k\mu \dfrac{-5U_i^j + 18U_{i+1}^j - 24U_{i+2}^j + 14U_{i+3}^j - 3U_{i+4}^j}{2h^3} - k\varepsilon U_i^j \dfrac{-3U_i^j + 4U_{i+1}^j - U_{i+2}^j}{2h} & ; \quad i = 1,2 \\ U_i^j - k\mu \dfrac{U_{i-3}^j - 8U_{i-2}^j + 13U_{i-1}^j - 13U_{i+1}^j + 8U_{i+2}^j - U_{i+3}^j}{8h^3} - k\varepsilon U_i^j \dfrac{-U_{i-1}^j + U_{i+1}^j}{2h} & ; \quad i = 3,4,\ldots,N-1 \\ U_i^j - k\mu \dfrac{3U_{i-4}^j - 14U_{i-3}^j + 24U_{i-2}^j - 18U_{i-1}^j + 5U_i^j}{2h^3} - k\varepsilon U_i^j \dfrac{U_{i-2}^j - 4U_{i-1}^j + 3U_i^j}{2h} & ; \quad i = N-2, N-1 \end{cases} \quad (5)$$

The solution $U(x,t)$ of the system (5) can be obtained by using the initial condition (2) and boundary conditions (3).

3. The Test Problems

The KdV equation is subject to following conditions in Problem 1 and problem 2 for single soliton and two soliton, respectively.

Problem 1. We consider KdV equation (1) with the boundary conditions

$$U(0,t) = U(2,t) = 0 \quad , \quad t > 0$$

and the initial condition

$$U(x,0) = 3C\,\mathrm{sech}^2(Ax + D)\ , \quad 0 \le x \le 2.$$

The exact solution of the problem 1 is given by [11] as

$$U(x,t) = 3C\,\mathrm{sech}^2(Ax - Bt + D)\ , \quad 0 \le x \le 2$$

where C and D are constants, $A = \sqrt{\varepsilon C/\mu}\big/2$ and $B = \varepsilon AC$.

Problem 2. We consider KdV equation (1) with the boundary conditions

$$U(0,t) = U(4,t) = 0 \quad , \quad t > 0$$

and the initial condition which will be derived from the exact solution [14] given as

$$U(x,t) = 12\mu(\log F)_{xx}\ , \quad 0 \le x \le 4$$

where

$$F = 1 + \exp(\eta_1) + \exp(\eta_2) + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 \exp(\eta_1 + \eta_2),$$

$\eta_i = \alpha_i x - \alpha_i^3 \mu t + bi$, $(i = 1,2)$,

$\alpha_1 = \sqrt{\frac{0.3}{\mu}}$, $\alpha_2 = \sqrt{\frac{0.1}{\mu}}$, $b_1 = -0.48\alpha_1$, $b_2 = -1.07\alpha_2$.

4. Numerical Results and Discussion

Method of lines (MOL) solution of the KdV equation with the initial and the boundary conditions for the small times was presented. We replaced the partial derivatives, U_x and U_{xxx}, in KdV equation with algebraic approximations evaluated at knots. The MOL approach leads to a set of algebraic equations. Algebraic equations were solved by using Euler's method. The performance of the method was tested on known two model problems.

In the both problems all computations were accomplished by taking the parameters $\mu = 4.84 \times 10^{-4}$, $\varepsilon = 1.0$, $D = -6.0$, $C = 0.3$. The method of lines solutions were compared with their exact solutions and other numerical solutions in literature [1,4,6]. The numerical solutions in Ref [1,4,6] were obtained by employing Galerkin finite element method (GFEM), quadratic spline approximation technique (QSAT), classical explicit finite difference method (CFDM) and exponential finite difference method (EFDM). The comparisons were given in Tables 1-4. In the present method, calculations are more economical. Actually, the procedure in this study is the explicit finite difference method and the numerical results obtained are better the numerical solutions obtained by using classical finite difference method [1], Tables 1-4. On the other hand, to show the accuracy of presented method the percentage error is used and the results obtained were compared with the results in [4,6]. It was seen that they were in a good agreement, Table 5-6. Fig. 1-2 show that the MOL solutions of the problems satisfy the physical behavior of the problems. Consequently, the presented method performs well.

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Table List

Table 1. Comparison of numerical and exact solutions of the Problem 1 at $t = 0.005$

Table 2. Comparison of numerical and exact solutions of the Problem 1 at $t = 0.01$

Table 3. Comparison of numerical and exact solutions of the Problem 2 at $t = 0.005$

Table 4. Comparison of numerical and exact solutions of the Problem 2 at $t = 0.01$

Table 5. Percentage of the Problem 1 for some selected values of x

Table 6. Percentage of the Problem 2 for some selected values of x

Figure Captions

Figure 1 The MOL solution of the Problem 1 at $t = 0.01$

Figure 2 The MOL of the Problem 2 at $t = 0.01$

x	Numerical Solutions					Exact Solution
	Present	EFDM [1]	CFDM [1]	QSAT [4]	GFEM [6]	
0.0	0.0	0.0	0.0	0.00002131	0.0	0.0
0.1	0.00025687	0.00224711	0.00026665	0.00025688	0.00025692	0.00025688
0.2	0.00309192	0.00355567	0.00320906	0.00309233	0.00309265	0.00309232
0.3	0.03657743	0.03777912	0.03794673	0.03658499	0.03658902	0.03658530
0.4	0.35579513	0.35698115	0.36617631	0.35572566	0.35578320	0.35574620
0.5	0.86266891	0.86070242	0.85627247	0.86304812	0.86299040	0.86305602
0.6	0.17787845	0.17729622	0.17201488	0.17787816	0.17789270	0.17787419
0.7	0.01626957	0.01512775	0.01568136	0.01627113	0.01627284	0.01627119
0.8	0.00136062	0.00113721	0.00131132	0.00136082	0.00136096	0.00136083
0.9	0.00011293	0.00009312	0.00010884	0.00011294	0.00011295	0.00011294
1.0	0.00000937	0.00000779	0.00000903	0.00000937	0.00000937	0.00000937
1.1	0.00000078	0.00000065	0.00000075	0.00000078	0.00000078	0.00000078
1.2	0.00000006	0.00000005	0.00000006	0.00000006	0.00000006	0.00000006
1.3	0.00000001	0.0	0.00000001	0.00000001	0.00000001	0.00000001
1.4	0.0	0.0	0.0	0.0	0.0	0.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 1. Comparison of numerical and exact solutions of the Problem 1 at $t = 0.005$

x	Numerical Solutions					Exact Solution
	Present	EFDM [1]	CFDM [1]	QSAT [4]	GFEM [6]	
0.0	0.0	0.0	0.0	0.00002053	0.0	0.0
0.1	0.00025905	0.00405629	0.00026665	0.00024748	0.00024750	0.00024746
0.2	0.00297840	0.00389535	0.00320906	0.00297935	0.00297987	0.00297915
0.3	0.03525619	0.03764657	0.03794673	0.03527067	0.03527915	0.03527078
0.4	0.34559661	0.34792815	0.36616631	0.34543282	0.34553080	0.34551645
0.5	0.86858195	0.86481172	0.85627247	0.86929925	0.86920580	0.86931946
0.6	0.18390738	0.18273885	0.17201488	0.18392550	0.18394800	0.18391215
0.7	0.01688133	0.01458343	0.01568136	0.01688369	0.01688850	0.01688448
0.8	0.00141213	0.00095558	0.00131132	0.00141248	0.00141291	0.00141257
0.9	0.00011720	0.00007766	0.00010884	0.00011723	0.00011727	0.00011724
1.0	0.00000972	0.00000672	0.00000903	0.00000972	0.00000972	0.00000972
1.1	0.00000081	0.00000056	0.00000075	0.00000081	0.00000080	0.00000081
1.2	0.00000007	0.00000005	0.00000006	0.00000007	0.00000007	0.00000007
1.3	0.00000001	0.0	0.00000001	0.00000001	0.00000001	0.00000001
1.4	0.0	0.0	0.0	0.0	0.0	0.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 2. Comparison of numerical and exact solutions of the Problem 1 at $t = 0.01$

x	Numerical Solutions					Exact Solution
	Present	EFDM [1]	CFDM [1]	QSAT [4]	GFEM [6]	
0.0	0.0	0.0	0.0	0.00000983	0.0	0.0
0.2	0.00141320	0.00161957	0.00130445	0.00141347	0.00141361	0.00141318
0.4	0.16389664	0.16250630	0.16402277	0.16386027	0.16388340	0.16055731
0.6	0.07311703	0.07258729	0.07123047	0.07310281	0.07309954	0.07392886
0.8	0.00132831	0.00124599	0.00137135	0.00132840	0.00132846	0.00132844
1.0	0.01291963	0.01296958	0.01290987	0.01291970	0.01291965	0.01290512
1.2	0.11307360	0.11226482	0.11251313	0.11307170	0.11307170	0.11263312
1.4	0.05037501	0.05036881	0.05013744	0.05037528	0.05037538	0.05056181
1.6	0.00354260	0.00348081	0.00353736	0.00354259	0.00354259	0.00354377
1.8	0.00020257	0.00019821	0.00020283	0.00020257	0.00020257	0.00020258
2.0	0.00001144	0.00001119	0.00001146	0.00001144	0.00001144	0.00001144
2.2	0.00000065	0.00000063	0.00000065	0.00000064	0.00000065	0.00000065
2.4	0.00000004	0.00000004	0.00000004	0.00000004	0.00000004	0.00000004
2.6	0.0	0.00000001	0.00000001	0.0	0.0	0.0
2.8	0.0	0.0	0.0	0.0	0.0	0.0
⋮	0.0	⋮	⋮	⋮	⋮	⋮
4.0	⋮	0.0	0.0	0.0	0.0	0.0

Table 3. Comparison of numerical and exact solutions of the Problem 2 at $t = 0.005$

x	Numerical Solutions					Exact Solution
	Present	EFDM [1]	CFDM [1]	QSAT [4]	GFEM [6]	
0.0	0.0	0.0	0.0	0.00000948	0.0	0.0
0.2	0.00136165	0.00177650	0.00114718	0.00136205	0.00136239	0.00136152
0.4	0.16263504	0.16032857	0.16335221	0.16248533	0.16276950	0.15599867
0.6	0.07468018	0.07393146	0.07119635	0.07464120	0.07465229	0.07645024
0.8	0.00134414	0.00118044	0.00143272	0.00134430	0.00134446	0.00134442
1.0	0.01284605	0.01298586	0.01286635	0.01284589	0.01284584	0.01281754
1.2	0.11322100	0.11194221	0.11244069	0.11321707	0.11321610	0.11233428
1.4	0.05047206	0.05061504	0.05015009	0.05047149	0.05047162	0.05084661
1.6	0.00356638	0.00345329	0.00356666	0.00356654	0.00356654	0.00356898
1.8	0.00020403	0.00019593	0.00020517	0.00020402	0.00020403	0.00020404
2.0	0.00001152	0.00001106	0.00001159	0.00001152	0.00001152	0.00001152
2.2	0.00000065	0.00000062	0.00000065	0.00000065	0.00000065	0.00000065
2.4	0.00000004	0.00000004	0.00000004	0.00000004	0.00000004	0.00000004
2.6	0.0	0.0	0.0	0.00000001	0.0	0.0
2.8	0.0	0.0	0.0	0.0	0.0	0.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
4.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 4. Comparison of numerical and exact solutions of the Problem 2 at $t = 0.01$

t		*x*=0.2	*x*=0.4	*x*=0.6	*x*=0.8	*x*=1.0
0.005	Present	0.0129	0.0137	0.0024	0.0154	0.0
	GFEM [6]	0.0107	0.0104	0.0104	0.0096	0.0
	QSAT [4]	0.0003	0.0057	0.0022	0.0007	0.0
0.01	Present	0.0251	0.0232	0.0026	0.0311	0.0
	GFEM [6]	0.0242	0.0042	0.0195	0.0240	0.0
	QSAT [4]	0.0067	0.0242	0.0072	0.0063	0.0

Table 5. Percentage of the Problem 1 for some selected values of *x*

t		*x*=0.4	*x*=0.8	*x*=1.2	*x*=1.6	*x*=2.0
0.005	Present	2.0798	0.0098	0.3910	0.0330	0.0
	GFEM [6]	2.0295	0.0015	0.3879	0.0333	0.0
	QSAT [4]	2.0572	0.0030	0.3894	0.0336	0.0
0.01	Present	4.2541	0.0208	0.7893	0.0728	0.0
	GFEM [6]	4.3403	0.0030	0.7850	0.0684	0.0
	QSAT [4]	4.1581	0.0089	0.7859	0.0684	0.0

Table 6. Percentage of the Problem 2 for some selected values of *x*

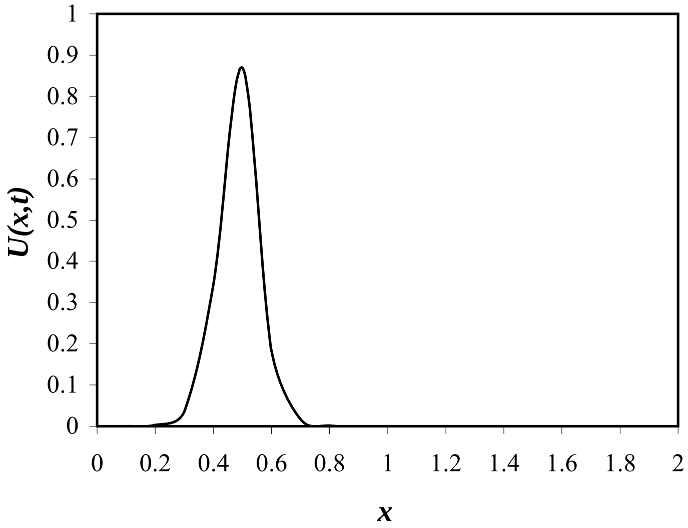

Fig. 1 The MOL solution of the Problem 1 at $t = 0.01$

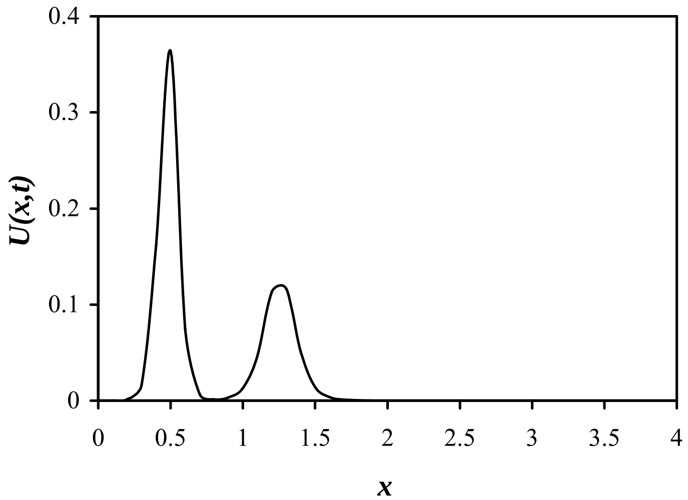

Fig. 2 The MOL of the Problem 2 at $t = 0.01$

Received: May 26, 2006