A remark on the existence of positive solutions for a reaction-diffusion system

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Abstract
We consider the existence of positive solutions for the reaction-diffusion system

\[
\begin{align*}
-\Delta u &= \lambda v^{\alpha}, \quad x \in \Omega, \\
-\Delta v &= \lambda w^{\beta}, \quad x \in \Omega, \\
-\Delta w &= \lambda u^{\gamma}, \quad x \in \Omega, \\
u &= v = w = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \lambda \) is a positive parameter, \( \Delta \) is the Laplacian operator, \( \alpha, \beta, \gamma > 0 \), and \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N > 1) \) with smooth boundary \( \partial \Omega \). We prove the existence of positive solution for each \( \lambda > 0 \). We establish our results by using the method of sub-super solutions.

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1 Introduction

In this paper we consider the existence of positive solutions for the quasi-linear reaction-diffusion system of the form

\[
\begin{align*}
-\Delta u &= \lambda v^{\alpha}, \quad x \in \Omega \\
-\Delta v &= \lambda w^{\beta}, \quad x \in \Omega \\
-\Delta w &= \lambda u^{\gamma}, \quad x \in \Omega \\
u &= v = w = 0, \quad x \in \partial \Omega,
\end{align*}
\]
where $\lambda$ is a positive parameter, $\Delta$ is the Laplacian operator, $\alpha, \beta, \gamma > 0$, and $\Omega$ is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial \Omega$. In recent years, many authors have investigated the following initial boundary value problem of a class of quasilinear reaction-diffusion system

$$\begin{cases}
    u_t = \Delta u + v^\alpha, \\
    v_t = \Delta v + w^\beta, \\
    w_t = \Delta w + u^\gamma,
\end{cases} \quad (x,t) \in \Omega \times (0,T), \tag{2}$$

where $\Omega$ is as above. Yang and Lu [7] studied the nonexistence of positive solutions to the system (2).

Systems of the form (1) arise in several context in biology and engineering (see [5]). It provides a simple model to describe, for instance, the interaction of three diffusing biological species. $u, v$ and $w$ represent the densities of three species. See [6] for details on the physical models involving more general reaction-diffusion system.

In this short paper, we shall prove that if $\alpha, \beta < 1$ and $\gamma < 1$, (1) admits a positive solution for each $\lambda > 0$. Our approach is based on the method of sub- and supersolutions, see [3]. We refer to [1, 2, 4] for additional results on elliptic systems.

2 Existence results

To prove our existence results we use the method of sub-super solutions. To do so, we now define sub and super solutions of (1).

**Definition 2.1.** A pair of nonnegative functions $(\psi_1, \psi_2, \psi_3), (z_1, z_2, z_3)$ in $C_0^2(\bar{\Omega}) \times C_0^2(\bar{\Omega}) \times C_0^2(\bar{\Omega})$ are called a subsolution and supersolution of (1) if they satisfy $\psi_i(x) \leq z_i(x)$ in $\Omega$ for $i = 1, 2, 3$, and

$$-\Delta \psi_1 \leq \lambda \psi_2^\alpha, \quad -\Delta \psi_2 \leq \lambda \psi_3^\beta, \quad -\Delta \psi_3 \leq \lambda \psi_1^\gamma, \quad x \in \Omega,$$

and

$$-\Delta z_1 \geq \lambda z_2^\alpha, \quad -\Delta z_2 \geq \lambda z_3^\beta, \quad -\Delta z_3 \geq \lambda z_1^\gamma, \quad x \in \Omega.$$

We shall obtain the existence of positive solution to system (1) by constructing a positive subsolution $(\psi_1, \psi_2, \psi_3)$ and supersolution $(z_1, z_2, z_3)$.

Our main result is formulate in the following theorem.
Theorem 2.2. Let \( \alpha, \beta, \gamma > 0, \alpha, \beta < 1 \) and \( \gamma < 1 \). Then system (1) has a positive solution for each \( \lambda > 0 \).

Proof. Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta\) with Dirichlet boundary conditions and \( \phi_1 \) denote the corresponding eigenfunction, satisfying \( \phi_1(x) > 0 \) in \( \Omega \), \( |\nabla \phi_1| > 0 \) on \( \partial \Omega \) and \( ||\phi_1||_\infty = 1 \). We shall verify that \((\psi_1, \psi_2, \psi_3) = (\psi, \psi, \psi)\), where \( \psi = \frac{k}{\alpha} \phi_1^2 \), is a subsolution of (1), where \( k > 0 \) is small and specified later. A calculation shows that

\[
-\Delta \psi = - \frac{k}{2} \Delta \phi_1^2 \\
= -k (|\nabla \phi_1|^2 + \phi_1 \Delta \phi_1) \\
= k (\lambda_1 \phi_1^2 - |\nabla \phi_1|^2).
\]

Since \( \phi_1 = 0 \) and \( |\nabla \phi_1| > 0 \) on \( \partial \Omega \), there is \( \delta > 0 \) such that

\[
\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq 0, \quad x \in \Omega_\delta,
\]

with \( \Omega_\delta = \{x \in \Omega \mid d(x, \partial \Omega) \leq \delta\} \). Which implies that

\[
k (\lambda_1 \phi_1^2 - |\nabla \phi_1|^2) \leq 0 \leq \lambda \psi^\alpha, \quad x \in \Omega_\delta,
\]

Next, we note that \( \phi_1(x) \geq \eta > 0 \) in \( \Omega_0 = \Omega \setminus \Omega_\delta \) for some \( \eta > 0 \). Since \( \alpha < 1 \) and, then there is \( k_0 > 0 \) such that if \( k \in (0, k_0) \) we have

\[
k^{1-\alpha} \lambda_1 \phi_1^2 \leq \lambda \left(\frac{1}{2}\right)^\alpha \eta^{2\alpha} \leq \lambda \left(\frac{1}{2}\right)^\alpha \phi_1^{2\alpha}, \quad x \in \Omega_0,
\]

\[
k^{1-\beta} \lambda_1 \phi_1^2 \leq \lambda \left(\frac{1}{2}\right)^\beta \eta^{2\beta} \leq \lambda \left(\frac{1}{2}\right)^\beta \phi_1^{2\beta}, \quad x \in \Omega_0,
\]

and

\[
k^{1-\gamma} \lambda_1 \phi_1^2 \leq \lambda \left(\frac{1}{2}\right)^\gamma \eta^{2\gamma} \leq \lambda \left(\frac{1}{2}\right)^\gamma \phi_1^{2\gamma}, \quad x \in \Omega_0.
\]

Hence

\[
-\Delta \psi = k (\lambda_1 \phi_1^2 - |\nabla \phi_1|^2) \\
\leq \lambda \psi^\alpha, \quad x \in \Omega_0.
\]
Thus

\[-\Delta \psi \leq \lambda \psi^\alpha, \quad x \in \Omega.\]

Similarly, we have

\[-\Delta \psi \leq \lambda \psi^\beta, \quad x \in \Omega,\]

and

\[-\Delta \psi \leq \lambda \psi^\gamma, \quad x \in \Omega.\]

i.e. \((\psi, \psi, \psi)\) is a subsolution of (1).

Next, let \(\zeta(x)\) be the positive solutions, of the problem

\[
\begin{cases}
-\Delta \zeta = 1, & x \in \Omega, \\
\zeta = 0, & x \in \partial \Omega.
\end{cases}
\]

Let

\((z_1, z_2, z_3) = (C_1 \zeta, C_2 \zeta, C_3 \zeta),\)

where \(C_1, C_2, C_3 > 0\) are large numbers to be chosen later. We shall verify that \((z_1, z_2, z_3)\) is a supersolution of (1). A calculation shows that

\[-\Delta z_1 = C_1.\]

Similarly we have

\[-\Delta z_2 = C_2, \quad -\Delta z_3 = C_3.\]

Let \(l = ||\zeta||_\infty\), it is easy to prove that there exist positive large constants \(C_1, C_2, C_3\) such that

\[C_1 \geq \lambda(C_2 l)^\alpha, \quad C_2 \geq \lambda(C_3 l)^\beta, \quad C_3 \geq \lambda(C_1 l)^\gamma.\]

Then we have
Existence of positive solutions

\[
C_1 = \lambda (C_2 l)^\alpha \\
\geq \lambda (C_2 \zeta)^\alpha \\
\geq \lambda (z_2)^\alpha,
\]
similarly we have

\[
C_2 \geq \lambda z_3^\beta, \quad C_3 \geq \lambda z_1^\gamma,
\]
and therefore

\[
-\Delta z_1 \geq \lambda z_2^\alpha, \quad -\Delta z_2 \geq \lambda z_3^\beta,
\]
and

\[
-\Delta z_3 \geq \lambda z_1^\gamma,
\]
i.e. \((z_1, z_2, z_3)\) is a supersolution of (1) with \(z_i \geq \psi_i\) in \(\Omega\) for large \(C_1, C_2, C_3, i = 1, 2, 3\). Thus, by the comparison principle, there exists a solution \((u, v, w)\) of (1) with \(\psi_1 \leq u \leq z_1, \quad \psi_2 \leq v \leq z_2, \quad \psi_3 \leq v \leq z_3\). This completes the proof of Theorem 2.2.

References


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