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A remark on the existence of positive solutions for a reaction-diffusion system

G.A. Afrouzi

Department of Mathematics, Faculty of Basic Sciences,
Mazandaran University, Babolsar, Iran
e-mail: afrouzi@umz.ac.ir

S.H. Rasouli

Department of Mathematics, Faculty of Basic Sciences,
Mazandaran University, Babolsar, Iran
e-mail: s.h.rasouli@umz.ac.ir

Abstract

We consider the existence of positive solutions for the reaction-diffusion system

$$\begin{cases} -\Delta u = \lambda v^{\alpha}, & x \in \Omega, \\ -\Delta v = \lambda w^{\beta}, & x \in \Omega, \\ -\Delta w = \lambda u^{\gamma}, & x \in \Omega, \\ u = v = w = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, Δ is the Laplacian operator, α, β, $\gamma > 0$, and Ω is a bounded domain in $R^N (N > 1)$ with smooth boundary $\partial\Omega$. We prove the existence of positive solution for each $\lambda > 0$. We establish our results by using the method of sub-super solutions.

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1 Introduction

In this paper we consider the existence of positive solutions for the quasilinear reaction-diffusion system of the form

$$\begin{cases} -\Delta u = \lambda v^{\alpha}, & x \in \Omega \\ -\Delta v = \lambda w^{\beta}, & x \in \Omega \\ -\Delta w = \lambda u^{\gamma}, & x \in \Omega \\ u = v = w = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where λ is a positive parameter, Δ is the Laplacian operator, α, β, $\gamma > 0$, and Ω is a bounded domain in $R^N (N > 1)$ with smooth boundary $\partial\Omega$.

In recent years, many authors have investigated the following initial boundary value problem of a class of quasilinear reaction-diffusion system

$$\begin{cases} u_t = \Delta u + v^\alpha, \\ v_t = \Delta v + w^\beta, \\ w_t = \Delta w + u^\gamma, \qquad (x,t) \in \Omega \times (0,T), \end{cases} \tag{2}$$

where Ω is as above. Yang and Lu [7] studied the nonexistence of positive solutions to the system (2).

Systems of the form (1) arise in several context in biology and engineering (see [5]). It provides a simple model to describe, for instance, the interaction of three diffusing biological species. u, v and w represent the densities of three species. See [6] for details on the physical models involving more general reaction-diffusion system.

In this short paper, we shall prove that if $\alpha, \beta < 1$ and $\gamma < 1$, (1) admits a positive solution for each $\lambda > 0$. Our approach is based on the method of sub- and supersolutions, see [3]. We refer to [1, 2, 4] for additional results on elliptic systems.

2 Existence results

To prove our existence results we use the method of sub-super solutions. To do so, we now define sub and super solutions of (1).

Definition 2.1. A pair of nonnegative functions (ψ_1, ψ_2, ψ_3), (z_1, z_2, z_3) in $C_0^2(\bar{\Omega}) \times C_0^2(\bar{\Omega}) \times C_0^2(\bar{\Omega})$ are called a subsolution and supersolution of (1) if they satisfy $\psi_i(x) \le z_i(x)$ in Ω for $i = 1, 2, 3$, and

$$-\Delta\psi_1 \le \lambda\,\psi_2^\alpha, \ -\Delta\psi_2 \le \lambda\,\psi_3^\beta, \ -\Delta\psi_3 \le \lambda\,\psi_1^\gamma, \qquad x \in \Omega,$$

and

$$-\Delta z_1 \ge \lambda\, z_2^\alpha, \ -\Delta z_2 \ge \lambda\, z_3^\beta, \ -\Delta z_3 \ge \lambda\, z_1^\gamma, \qquad x \in \Omega.$$

We shall obtain the existence of positive solution to system (1) by constructing a positive subsolution (ψ_1, ψ_2, ψ_3) and supersolution (z_1, z_2, z_3).

Our main result is formulate in the following theorem.

Theorem 2.2. Let α, β, $\gamma > 0$, $\alpha, \beta < 1$ and $\gamma < 1$. Then system (1) has a positive solution for each $\lambda > 0$.

Proof. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 denote the corresponding eigenfunction, satisfying $\phi_1(x) > 0$ in Ω, $|\nabla\phi_1| > 0$ on $\partial\Omega$ and $||\phi_1||_\infty = 1$. We shall verify that $(\psi_1, \psi_2, \psi_3) = (\psi, \psi, \psi)$, where $\psi = \frac{k}{2}\phi_1^2$, is a subsolution of (1), where $k > 0$ is small and specified later. A calculation shows that

$$\begin{aligned}
-\Delta\psi &= -\frac{k}{2}\,\Delta\phi_1^2\\
&= -k\,(|\nabla\phi_1|^2 + \phi_1\,\Delta\phi_1)\\
&= k\,(\lambda_1\phi_1^2 - |\nabla\phi_1|^2).
\end{aligned}$$

Since $\phi_1 = 0$ and $|\nabla\phi_1| > 0$ on $\partial\Omega$, there is $\delta > 0$ such that

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le 0, \qquad x \in \bar{\Omega}_\delta,$$

with $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. Which implies that

$$k\,(\lambda_1\phi_1^2 - |\nabla\phi_1|^2) \;\le\; 0 \;\le\; \lambda\,\psi^\alpha, \qquad x \in \bar{\Omega}_\delta,$$

Next, we note that $\phi_1(x) \ge \eta > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\delta$ for some $\eta > 0$. Since $\alpha < 1$ and, then there is $k_0 > 0$ such that if $k \in (0, k_0)$ we have

$$k^{1-\alpha}\,\lambda_1\,\phi_1^2 \;\le\; \lambda\,(\frac{1}{2})^\alpha\,\eta^{2\alpha} \le\; \lambda\,(\frac{1}{2})^\alpha\,\phi_1^{2\alpha}, \qquad x \in \Omega_0,$$

$$k^{1-\beta}\,\lambda_1\,\phi_1^2 \;\le\; \lambda\,(\frac{1}{2})^\beta\,\eta^{2\beta} \le\; \lambda\,(\frac{1}{2})^\beta\,\phi_1^{2\beta}, \qquad x \in \Omega_0,$$

and

$$k^{1-\gamma}\,\lambda_1\,\phi_1^2 \;\le\; \lambda\,(\frac{1}{2})^\gamma\,\eta^{2\gamma} \le\; \lambda\,(\frac{1}{2})^\gamma\,\phi_1^{2\gamma}, \qquad x \in \Omega_0.$$

Hence

$$\begin{aligned}
-\Delta\psi &= k\,(\lambda_1\phi_1^2 - |\nabla\phi_1|^2)\\
&\le \lambda\,\psi^\alpha, \qquad x \in \Omega_0.
\end{aligned}$$

Thus

$$-\Delta\psi \,\le\, \lambda\,\psi^{\alpha}, \qquad x \in \Omega.$$

Similarly, we have

$$-\Delta\psi \,\le\, \lambda\,\psi^{\beta}, \qquad x \in \Omega,$$

and

$$-\Delta\psi \,\le\, \lambda\,\psi^{\gamma}, \qquad x \in \Omega.$$

i.e. (ψ, ψ, ψ) is a subsolution of (1).

Next, let $\zeta(x)$ be the positive solutions, of the problem

$$\begin{cases} -\Delta\,\zeta = 1, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega. \end{cases}$$

Let

$$(z_1, z_2, z_3) = (C_1\,\zeta,\, C_2\,\zeta,\, C_3\,\zeta),$$

where C_1, C_2, $C_3 > 0$ are large numbers to be chosen later. We shall verify that (z_1, z_2, z_3) is a supersolution of (1). A calculation shows that

$$-\Delta z_1 = C_1.$$

Similarly we have

$$-\Delta z_2 = C_2, \qquad -\Delta z_3 = C_3.$$

Let $l = ||\zeta||_{\infty}$, it is easy to prove that there exist positive large constants C_1, C_2, C_3 such that

$$C_1 \ge \lambda(C_2\,l)^{\alpha}, \quad C_2 \ge \lambda(C_3\,l)^{\beta}, \quad C_3 \ge \lambda(C_1\,l)^{\gamma}.$$

Then we have

$$\begin{aligned} C_1 &= \lambda(C_2\, l)^\alpha \\ &\geq \lambda(C_2\zeta)^\alpha \\ &\geq \lambda(z_2)^\alpha, \end{aligned}$$

similarly we have

$$C_2 \geq \lambda z_3^\beta, \quad C_3 \geq \lambda z_1^\gamma,$$

and therefore

$$-\Delta z_1 \geq \lambda z_2^\alpha, \quad -\Delta z_2 \geq \lambda z_3^\beta,$$

and

$$-\Delta z_3 \geq \lambda z_1^\gamma,$$

i.e. (z_1, z_2, z_3) is a supersolution of (1) with $z_i \geq \psi_i$ in Ω for large C_1, C_2, C_3, $i = 1, 2, 3$. Thus, by the comparison principle, there exists a solution (u, v, w) of (1) with $\psi_1 \leq u \leq z_1, \quad \psi_2 \leq v \leq z_2, \quad \psi_3 \leq v \leq z_3$. This completes the proof of Theorem 2.2.

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