

Conditions for Hypercyclicity Criterion

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Abstract

We give necessary and sufficient conditions for an operator on a separable Hilbert space satisfies the Hypercyclicity Criterion.

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1. Introduction

Suppose that X is a separable topological vector space and T is a continuous linear mapping on X . If $x \in X$, then the orbit of x under T is denoted by $Orb(T, x)$ and is defined by $Orb(T, x) = \{x, Tx, T^2x, \dots\}$. An operator T is called hypercyclic if there is a vector x such that $Orb(T, x)$ is dense in X and in this case x is called a hypercyclic vector for T . In general the sequence $\{T_n : n \in \mathbb{N}\}$ of bounded linear operators on X is called hypercyclic if the set $\{T_n x : n \in \mathbb{N}\}$ is dense in X for some vector $x \in X$.

It is interesting to know that many continuous linear mappings can actually be hypercyclic. The first example of a hypercyclic operator appeared in the space of entire functions, by Birkhoff [2] in 1929. He showed the hypercyclicity

of the translation operator. Also, in 1952, Maclane [11] proved the hypercyclicity of the differentiation operator. An example of a hypercyclic operator on Banach spaces was constructed by Rolewicz [12] in 1969. He showed that if B is the backward shift on ℓ^p , then λB is hypercyclic if and only if $|\lambda| > 1$.

A nice criterion namely Hypercyclicity Criterion, was developed independently by Kitai [10] and, Gethner and Shapiro [7]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [4], weighted shifts [14], adjoints of multiplication operators [5], and adjoints of subnormal and hyponormal operators [3].

This criterion is an important tool in much recent works on hypercyclic operators.

Theorem 1.1. (The Hypercyclicity Criterion). Suppose X is a separable Banach space and T is a continuous linear mapping on X . If there exist two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that:

1. $T^{n_k}y \rightarrow 0$ for every $y \in Y$,
2. There exist functions $S_{n_k} : Z \rightarrow X$ such that for every $z \in Z$, $S_{n_k}z \rightarrow 0$, and $T^{n_k}S_{n_k}z \rightarrow z$,

then T is hypercyclic.

If an operator T satisfies the hypothesis of Theorem 1.1, we say that T satisfies the Hypercyclicity Criterion with respect to the sequence $\{n_k\}$.

Note that the sequence $\{n_k\}$ in Theorem 1.1, need not be the full sequence $\{n_k\} = \{k\}$ of positive integers. Salas [13] and Herrero [8] have shown that there are hypercyclic operators on Hilbert spaces that do not satisfy the Hypercyclicity Criterion for the full sequence, but so far no hypercyclic operator has been found that does not satisfy the Hypercyclicity Criterion in its general form.

In this paper we investigate the well-known question that says: Does every hypercyclic operator satisfies the hypothesis of the Hypercyclicity Criterion? (See[1], [15]).

The following question was raised by Herrero in [9].

Question 1. (Herrero's question). Is $T \oplus T$ hypercyclic whenever T is hypercyclic?

Bes and Peris proved that Herrero's question is equivalent to the following question([1]):

Question 2. (Necessity of the Hypercyclicity Criterion). If T is hypercyclic ,does T satisfies the hypothesis of the Hypercyclicity Criterion?

Theorem 1.2 ([1]). T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic.

In the present paper we will give necessary and sufficient conditions, in terms of open subsets, for an operator on a separable Frechet space satisfies the Hypercyclicity Criterion.

2. Main Results

From now on we suppose that H is a separable infinite dimensional Frechet space. The following result is the main tool to prove the hypercyclicity of an operator (see[7]).

Proposition 2.1. If X is a separable Frechet space, then the sequence $\{T_n : n \in \mathbb{N}\}$ of bounded linear operators on X is hypercyclic if and only if for any two non-void open sets U and V in X , $T_n U \cap V \neq \emptyset$ for infinite number of positive integers n .

The following theorem states the Hypercyclicity Criterion in terms of open sets.

Theorem 2.2. For any operator $T \in B(X)$, the following conditions are equivalent:

- i)* T satisfies the Hypercyclicity Criterion.
- ii)* T is hypercyclic and for each non-void open subset U and each neigh-

neighborhood W of zero, $T^n U \cap W \neq \phi$ and $T^{-n} U \cap W \neq \phi$ for some integer n .

iii) For each pair U and V of non-void open subsets of X , and each neighborhood W of zero, $T^n U \cap W \neq \phi$ and $T^n W \cap V \neq \phi$ for some integer n .

Proof. It is easy to see that (i) implies (ii). Suppose that T satisfies the condition (ii), U and V are non-void open subsets of X and W is a neighborhood of zero. Since T is hypercyclic, by Proposition 2.1 we have $U \cap T^{-m} V \neq \phi$ for some positive integer m . Now let G be a neighborhood of zero that is contained in $W \cap T^{-m} W$. By condition (ii), there exists some positive integer n such that $T^{-n} G \cap (U \cap T^{-m} V) \neq \phi$ and $G \cap T^{-n} (U \cap T^{-m} V) \neq \phi$. But $T^{-n} G \cap (U \cap T^{-m} V)$ is a subset of $T^{-n} W \cap U$, hence $T^{-n} W \cap U \neq \phi$. Also $G \cap T^{-n} (U \cap T^{-m} V)$ is a subset of $T^{-m} W \cap T^{-n} (T^{-m} V) = T^{-m} (W \cap T^{-n} V)$, thus $T^{-n} V \cap W \neq \phi$ which satisfies the condition (iii).

Now we prove that (iii) implies (i). By Theorem 1.2 it is sufficient to prove that $T \oplus T$ is hypercyclic. For this consider four arbitrary open subsets U_i and V_i for $i = 1, 2$. There exist open subsets \hat{U}_i and \hat{V}_i for $i = 1, 2$, and a neighborhood W_\circ of zero such that :

$$\hat{U}_i + W_\circ \subseteq U_i; \quad \hat{V}_i + W_\circ \subseteq V_i; \quad i = 1, 2.$$

Note that condition (iii) implies that T is hypercyclic. Hence there exist integers p and q such that:

$$G_1 = \hat{U}_1 \cap T^{-p} \hat{V}_1 \neq \phi; \quad G_2 = \hat{U}_2 \cap T^{-q} \hat{V}_2 \neq \phi.$$

Put

$$W = W_\circ \cap T^{-p} W_\circ \cap T^{-q} W_\circ.$$

Now by condition (iii) there is an integer n such that:

$$T^n G_1 \cap W \neq \phi; \quad T^n W \cap G_2 \neq \phi.$$

Choose the vectors x_o and y_o in X such that $x_o \in \widehat{U}_1$, $T^p x_o \in \widehat{V}_1$, $T^n x_o \in W$, $y_o \in W$, $T^n y_o \in \widehat{U}_2$ and $T^{n+q} y_o \in \widehat{V}_2$. Put $x = x_o + y_o$ and $y = T^p x_o + T^q y_o$. Then $x \oplus y \in U_1 \oplus V_1$ and $(T \oplus T)^n(x \oplus y) \in U_2 \oplus V_2$. So $T \oplus T$ is hypercyclic and the proof is complete. \diamond

The following definition comes from [1].

Definition 2.3. Suppose that $T \in B(X)$ and $\{n_k\}$ is a sequence of positive integers. We say that T is hereditarily hypercyclic with respect to $\{n_k\}$ if for any subsequence $\{n_{k_m}\}$ of $\{n_k\}$, the sequence $\{T^{n_{k_m}}\}$ is hypercyclic.

In [1] the authors show that T satisfies the Hypercyclicity Criterion if and only if T is hereditarily hypercyclic. But we note that if T is hereditarily hypercyclic with respect to $\{n_k\}$, then T satisfies the Hypercyclicity Criterion with respect to some subsequence of $\{n_k\}$.

The following theorem gives a necessary and sufficient condition (in terms of open sets) for operators being hereditarily hypercyclic with respect to a sequence $\{n_k\}$.

Proposition 2.4. For any operator $T \in B(X)$ and any sequence $\{n_k\}$ of integers, the following conditions are equivalent:

- i) T is hereditarily hypercyclic with respect to $\{n_k\}$.
- ii) For each pair U, V of non-void open subsets of X , there is $N \geq 1$ such that $T^{n_k} U \cap V \neq \phi$ for any $k \geq N$.

Proof. (i) \rightarrow (ii) : Suppose that the condition (ii) does not hold. So there exist some pair U, V of non-void open sets such that $T^{n_{k_m}} U \cap V = \phi$ for some subsequence $\{n_{k_m}\}$ of $\{n_k\}$. This is a contradiction since $\{T^{n_{k_m}}\}$ is hypercyclic. By a direct consequence of Proposition 2.1 the condition (ii) implies (i) and so the proof is complete. \diamond

Suppose that T is a bilateral weighted shift on $l^2(Z)$ with positive weights $\{w_j\}$. Let $\{e_j\}$ be the standard basis for $l^2(Z)$. Then T acts on this basis

elements by $Te_j = w_j e_{j-1}$ for all $j \in Z$. We may define a right inverse S of T as $Se_j = \frac{1}{w_{j+1}} e_{j+1}$.

The following theorem is well known and was proved in [13]. Here we will give another proof that is independent from the proof of [13].

Theorem 2.5. Let $\{n_k\} \subseteq \mathbb{N}$ and $T : l^2(Z) \rightarrow l^2(Z)$ be a unilateral weighted shift given by $Te_j = w_j e_{j-1}$.

Then the following conditions are equivalent:

- i) T satisfies the hypothesis of the Hypercyclicity Criterion with respect to $\{n_k\}$.
- ii) T is hereditarily hypercyclic with respect to $\{n_k\}$.
- iii) for all integers j we have $\lim_k \prod_{p=1}^{j+n_k} w_p = +\infty$ and $\lim_k \prod_{p=1}^{j+n_k} w_{-p} = 0$.

Proof For (i) \rightarrow (ii), see Theorem 2.3 in [1].

For the implication (ii) \rightarrow (iii), assume that T is hereditarily hypercyclic with respect to $\{n_k\}$. Also, let $j \in Z$ and $\delta \geq 0$. By Proposition 2.4, for some $x \in l^2(Z)$ there is an integer N such that :

$$\|x - e_{-j}\| \leq \delta \quad (1)$$

$$\|T^{n_k} k - e_j\| \leq \delta \quad \forall k \geq N \quad (2)$$

In terms of the coefficients of x , the inequality (1) implies that $|\langle x, e_j \rangle| \leq \delta$ and $|\langle x, e_{-j} \rangle| \geq 1 - \delta$. Now by the inequality (2) we get

$$\|T^{n_k}(\langle x, e_{-j} \rangle e_{-j})\| = \left(\prod_{p=j}^{j+n_k} w_{-p} \right) |\langle x, e_{-j} \rangle| \leq \delta.$$

Thus for any $k \geq N$ we have

$$\left| \prod_{p=j}^{j+n_k} w_{-p} \right| \leq \frac{\delta}{|\langle x, e_{-j} \rangle|} \leq \frac{\delta}{1 - \delta}.$$

On the other hand, inequality (2) implies that

$$\|T^{n_k}(\langle x, e_{j-n_k} \rangle e_{j-n_k}) - e_j\| \leq \delta$$

and consequently $\left| \left(\prod_{p=j}^{j+n_k} w_p \right) \langle x, e_j \rangle - 1 \right| \leq \delta$. Therefore we have

$$\left| \prod_{p=j}^{j+n_k} w_p \right| \geq \frac{1 - \delta}{|\langle x, e_j \rangle|} \leq \frac{1 - \delta}{\delta}$$

for any $k \geq N$. Hence $\lim_k \prod_{p=j}^{j+n_k} w_p = +\infty$ and $\lim_k \prod_{p=j}^{j+n_k} w_p = 0$, and so $w_{-p} = 0$, and

$$\lim_k \prod_{p=1}^{j+n_k} w_p = \infty$$

and

$$\lim_k \prod_{p=1}^{j+n_k} w_{-p} = 0.$$

Finally, we show that (iii) \rightarrow (i) : Suppose that the condition (iii) holds and let $Y = Z = \text{span}\{e_j : j \in Z\}$ and $S_{n_k} = S^{n_k}$, where S is the right inverse of T . So we get

$$T^{n_k} e_j = w_j w_{j-1} w_{j-2} \dots w_{j-n_k} = \prod_{p=j}^{j+n_k} w_{-p} \rightarrow 0$$

for all $j \in \mathbf{Z}$. Also

$$S_{n_k} e_j = S^{n_k} e_j = \frac{1}{w_j w_{j+1} w_{j+2} \dots w_{j+n_k}} = \frac{1}{\prod_{p=j}^{j+n_k} w_p} \rightarrow 0$$

and $T^{n_k} S_{n_k} e_j \rightarrow e_j$ for every $j \in Z$. Thus T satisfies the Hypercyclicity Criterion with respect to $\{n_k\}$ and so the proof is complete. \diamond

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