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On The β-Spherical Riesz Potential Generated By The $\beta-$Distance

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Abstract

In this paper, we studied the boundedness of the $\beta-$spherical Riesz potential generated by the $\beta-$distance from $L_\infty(S_\beta^{n-1})$ into $H^\lambda(S_\beta^{n-1})$.

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1 Introduction

It is well known that classical Riesz Potentials $I_\alpha\varphi = \varphi * |x|^{\alpha-n}$ are bounded operators from $L_p(R^n)$ to $L_q(R^n)$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 < \alpha < n$, $1 \leq p < q < \infty$ [1]. S.G. Samko and B.G. Vakulov have compared different approaches to function spaces of fractional smoothness on the unite sphere, constructed on the base of the space $C(S^{n-1})$, $n \geq 2$, the spaces $C^\lambda(S^{n-1})$ and the Hölder type spaces $H^\lambda(S^{n-1})$, $\lambda > 0$ were considered[2]. In this article we have defined the $\beta-$spherical Riesz potential generated by the $\beta-$distance and studied boundedness of this potential from $L_\infty(S_\beta^{n-1})$ into $H^\lambda(S_\beta^{n-1})$.

Suppose that S_β^{n-1} is the unite $\beta-$sphere of R^n $(n \geq 2)$ equipped with normalized Lebesque measure $d\sigma = d\sigma(.)$.

The $\beta-$distance between $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ points is defined by the following formula given in [3];

$$|x-y|_\beta := (|x_1-y_1|^{\frac{1}{\beta_1}} + |x_2-y_2|^{\frac{1}{\beta_2}} + ... + |x_n-y_n|^{\frac{1}{\beta_n}})^{\frac{|\beta|}{n}}.$$

where $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $\beta_k > 0$, $k = 1, 2, ..., n$, $|\beta| = \beta_1 + \beta_2 + ... + \beta_n$. Note that this distance has the following equality for any positive t,

$$\left(\left|t^{\beta_1}x_1\right|^{\frac{1}{\beta_1}} + ... + \left|t^{\beta_n}x_n\right|^{\frac{1}{\beta_n}}\right)^{\frac{|\beta|}{n}} = t^{\frac{|\beta|}{n}}\left|x\right|_\beta, \ \ t > 0.$$

This equality give us that non-isotropic $\beta-$distance is the order of a homogeneous function $\frac{|\beta|}{n}$. So the non-isotropic β-distance has the following properties:

1 $|x|_\beta = 0 \Leftrightarrow x = \theta$

2. $\left|t^\beta x\right|_\beta = |t|^{\frac{|\beta|}{n}} |x|_\beta$

3. $|x+y|_\beta \le 2^{\left(1+\frac{1}{\beta_{\min}}\right)\frac{|\beta|}{n}}(|x|_\beta + |y|_\beta)$.

Here we consider $\beta-$spherical coordinates by the following formulas :

$$x_1 = (\rho\cos\theta_1)^{2\beta_1}, ..., x_n = (\rho\sin\theta_1 \sin\theta_2 ... \sin\theta_{n-1})^{2\beta_n}.$$

For this we have $|x|_\beta = \rho^{\frac{2|\beta|}{n}}$. It can be seen that the Jacobian $J_\beta(\rho,\theta)$ of this transformation is $J_\beta(\rho,\theta) = \rho^{2|\beta|-1}\Omega_\beta(\theta)$, where $\Omega_\beta(\theta)$ is the bounded function which only depend on angles $\theta_1, \theta_2, ..., \theta_{n-1}$. It is clear that if $\beta_1 = \beta_2 = ... = \beta_n = \frac{1}{2}$, then the β-distance is the Euclidean distance.

We will define the $\beta-$spherical Riesz potential genrated by the $\beta-$distance as follow

$$I_\beta^\lambda f(x) = \int\limits_{S_\beta^{n-1}} |x-\sigma|_\beta^{\lambda-n+1} f(\sigma)d\sigma, \qquad 0<\lambda<1$$

where $x \in S_\beta^{n-1}$. For a positive r and any $x \in R^n$ we define the open $\beta-$ball $B_\beta(x,r)$ with radius r and a center x as

$$B_\beta(x,r) = \{\sigma : |x-\sigma|_\beta < r \ \}.$$

Definition. Let $0<\lambda<1$. The space $H^\lambda = H^\lambda([a,b])$ is defined as the space of continuous functions on $[a,b]$, such that

$$|f(x+h) - f(x)| \le M\,|h|^\lambda$$

for all $x,\ x+h \in [a,b]$.

Lemma 1. Let $0<\lambda<1$. Then,

$$\left| |x-\sigma|_\beta^{\lambda-n} - |\sigma-z|_\beta^{\lambda-n} \right| \le Mr\,|x-\sigma|_\beta^{\lambda-n-1}$$

where $\sigma \in R^n - B_\beta(x,2r)$ and M is a constant independing on x and σ.

Proof. Let $r = |x-z|_\beta$, $|x-\sigma|_\beta = a$, $|\sigma-z|_\beta = b$ and $a \neq 0,\ b \neq 0$ Then we have $0 < a-r < b < a+r$. Now we consider $f(t) = \frac{1}{t^\tau}$ where $t \in [a,b]$ [or $t \in [b,a]]$, $n-\lambda = \tau > 0$. Then function $f(t)$ has continuous and continuity derivatives in $[a,b]$ [or $[b,a]]$. Therefore, there is the following equality from Lagrange Theorem

$$|f(b)-f(a)| = \left|f^{'}(\xi)\right| |b-a| \quad \xi \in [a,b] \ [\text{or} \ \ \xi \in [b,a]]\,.$$

In this case $|b-a| < r$ we have the following inequality

$$\left|\frac{1}{b^\tau} - \frac{1}{a^\tau}\right| = \left|-\tau \frac{1}{\xi^{\tau+1}}\right| |b-a| \le \tau \left|\frac{1}{\xi^{\tau+1}}\right| r.$$

If $a < \xi < b$, then we have

$$\left|\frac{1}{b^\tau} - \frac{1}{a^\tau}\right| \le \tau \frac{1}{a^{\tau+1}} r \le Mr\,|x-y|_\beta^{\lambda-n-1}\,.$$

If $b < \xi < a, \ \xi \in (a-r,a), \ \xi = a - \theta r, \ 0 < \theta < 1$, then we have

$$\left|\frac{1}{b^\tau} - \frac{1}{a^\tau}\right| = \tau \frac{1}{(a-\theta r)^{\tau+1}} r \le Mr\,|x-y|_\beta^{\lambda-n-1}\,.$$

The proof is completed.

Lemma 2. The integral

$$J_\beta^{\lambda,\mu,\nu}(x,y) \quad = \quad \int\limits_{S_\beta^{n-1}} \frac{\left(|x-\sigma|_\beta + |y-\sigma|_\beta\right)^\lambda}{|x-\sigma|_\beta^\mu \, |y-\sigma|_\beta^\nu} d\sigma$$

where $x, y \in S_\beta^{n-1}$ and $0 < \mu < n-1, \ 0 < \nu < n-1, \ \lambda \in R^1$, admits the estimate

$$J_\beta^{\lambda,\mu,\nu}(x,y) \quad \le \quad M \begin{cases} |x-y|_\beta^{-\delta} & , \ \delta > 0 \\ \ln \frac{2}{|x-y|_\beta} & , \ \delta = 0 \\ 1 & , \ \delta < 0 \end{cases}$$

where $\delta = \frac{|\beta|}{n}(\mu+\nu-\lambda) + \beta - n$ and M is a constant independing on x and y.

Proof. In the proof of this Lemma, we follow Sobolev[4], where a similar statement was proved for regions in R^n. If we do the change of variables $\sigma = x - t\,|x-y|_\beta^\beta$, then we have

$$J_\beta^{\lambda,\mu,\nu}(x,y) \quad = \quad |x-y|_\beta^{-\delta} \int\limits_{S\left(\frac{x}{|x-y|_\beta^\beta}, \frac{1}{|x-y|_\beta^\beta}\right)} \frac{\left(|t|_\beta + |t-e|_\beta\right)^\lambda}{|t|_\beta^\mu \, |t-e|_\beta^\nu} dt$$

where $e=\frac{x-y}{|x-y|_\beta^\beta}$. We split the sphere $S(\frac{x}{|x-y|_\beta^\beta},\frac{1}{|x-y|_\beta^\beta})$ into two parts $S_1=B_\beta(0,2)\cap S(\frac{x}{|x-y|_\beta^\beta},\frac{1}{|x-y|_\beta^\beta})$ and $S_2=S(\frac{x}{|x-y|_\beta^\beta},\frac{1}{|x-y|_\beta^\beta})\backslash S_1$ where $B_\beta(0,2)$ is the $\beta-$ball of the radius 2 centered at the origin. In the represantation

$$\begin{aligned} J_\beta^{\lambda,\mu,\nu}(x,y) &= |x-y|_\beta^{-\delta}\left(\int\limits_{S_1} dt+\int\limits_{S_2} dt\right) \\ &= |x-y|_\beta^{-\delta}\left(J_\beta^1+J_\beta^2\right) \end{aligned}$$

the integral J_β^1 is bounded since the integrand is bounded beyond the singular points $t=0$ and $t=e$ at which the singularities are weak. In the integral J_β^2 the integrand is equivanlent to $|t|_\beta^{\lambda-\mu-\nu}$, so that

$$J_\beta^2 \leq M\int\limits_{S_2}|t|_\beta^{\lambda-\mu-\nu}\,dt.$$

Hence, after the inverse change of variables $t=\frac{\sigma}{|x-y|_\beta^\beta}-\frac{x}{|x-y|_\beta^\beta}$ we obtain

$$J_\beta^2 \leq M\,|x-y|_\beta^\delta\int\limits_{S_3}\frac{d\sigma}{|x-\sigma|_\beta^{\mu+\nu-\lambda}}$$

where $S_3(x,y)=\left\{\sigma:\ |\sigma|_\beta=1,\ |x-\sigma|_\beta\geq 2\,|x-y|_\beta\right\}$. The inequality is valid:

$$\Lambda_\beta(x,y) := \int\limits_{S_3}\frac{d\sigma}{|x-\sigma|_\beta^{n-1-\alpha}} \leq \begin{cases} |x-y|_\beta^\alpha & ,\ \alpha<n-1 \\ \ln\frac{2}{|x-y|_\beta} & ,\ \alpha=n-1 \\ 1 & ,\ \alpha>n-1 \end{cases}.$$

Theorem 1: The operator I_β^λ , $0<\lambda<1$, is bounded from $L_\infty(S_\beta^{n-1})$ into $H^\lambda(S_\beta^{n-1})$.

Proof. From Lemma 1 we have

$$\left|I_\beta^\lambda f(x)-I_\beta^\lambda f(y)\right| \leq M\,\|f\|_\infty\,M\,|x-y|_\beta\int\limits_{S_\beta^{n-1}}\frac{d\sigma}{|x-\sigma|_\beta^{n-1-\lambda}}.$$

Then the application of Lemma 2 completes the proof.

Corollary: $I_\beta^\lambda(C(S_\beta^{n-1}))\to H^\lambda(S_\beta^{n-1})$ and $I_\beta^\lambda(C(S_\beta^{n-1}))\neq H^\lambda(S_\beta^{n-1})$, $0<\lambda<1$.

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