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CURVES ON RULED SURFACES IN MINKOWSKI 3-SPACE

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Abstract

Shyuichi Izumiya and Nobuko Takeuchi [5] obtained some characterizations for Ruled surfaces. H.H.Hacısalihoğlu and Aysel Turgut [9] defined timelike Ruled surfaces and obtained some characterizations in timelike Ruled surfaces. Soon Meen Choi [3], Seoung Dal Jung and Jin Suk Pak [6] studied on Ruled surfaces.

In this paper, making use of method in paper of Shyuichi Izumiya and Nobuko Takeuchi we obtained some characterizations for timelike Ruled surfaces in Minkowski 3-space . Also we studied cylindrical helices and Bertrand curves from the view point of the theory of curves on timelike Ruled surfaces.

Some results in this paper clarify that the cylindrical helices are related to Gaussian curvature and the bertrand curve is related to mean curvature of timelike Ruled surfaces.

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1. Basic Notions and Properties

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ be a $3-$dimensional vector space, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$, the pseudo scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1+x_2y_2+x_3y_3$. We call $(\mathbb{R}^3, \langle,\rangle)$ a $3-$ dimensional pseudo Euclidean space or Minkowski $3-$ space. We denote $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle,\rangle)$.

We say that a vector x in $\mathbb{R}^3_1$ is spacelike, lightlike or timelike if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. The norm of the vector $x \in \mathbb{R}^3_1$ is defined by

$$\|x\| = \sqrt{|\langle x, x\rangle|}.$$

Let $\alpha : I \to \mathbb{R}^3_1$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a smooth regular curve in $\mathbb{R}^3_1$ ($i.e.$ $\alpha'(t) \neq 0$ for any $t \in I$), where I is an open interval. The curve α is called spacelike if $\langle \alpha', \alpha' \rangle > 0$ and timelike if $\langle \alpha', \alpha' \rangle < 0$ and lightlike if $\langle \alpha', \alpha' \rangle = 0$.

The arc-length of a spacelike curve α, measured from $\alpha(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \, dt.$$

Then the parameter s is determined such that $\|\alpha'(s)\| = 1$, where $\alpha'(s) = \frac{d\alpha}{ds}$. So we say that a spacelike curve α is parametrized by arc-length if it satisfies that $\|\alpha'(s)\| = 1$. Let us denote $t(s) = \alpha'(s)$ and we call $t(s)$ a unit tangent vector of α at s. We define the curvature by

$$\kappa(s) = \sqrt{|\langle \alpha''(s), \alpha''(s) \rangle|}.$$

If $\kappa(s) \neq 0$ then the unit principal normal vector n of a timelike curve α at s is given by $\alpha''(s) = \kappa(s).n(s)$.

For any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the pseudo-vector product of x and y is defined as follows:

$$x \Lambda y = (-(x_2 y_3 - x_3 y_2),\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1).$$

The unit vector $b(s) = t(s) \Lambda n(s)$ is called a unit binormal vector of the curve α at s [2, 7].

Let α be a timelike curve in $\mathbb{R}^3_1$ and let us denote $t(s) = \alpha'(s)$. Then we have the Frenet-Serret Formulae

$$\begin{aligned} t'(s) &= \kappa(s) n(s) \\ n'(s) &= \kappa(s) t(s) + \tau(s) b(s) \\ b'(s) &= -\tau(s) n(s), \end{aligned} \tag{1.1}$$

where $\tau(s)$ is the torsion of the curve α at s. For any unit speed timelike curve $\alpha : I \to \mathbb{R}^3_1$, we can define the Darboux vector field by

$$D(s) = -\tau(s) t(s) - \kappa(s) b(s). \tag{1.2}$$

We define a vector field

$$\widetilde{D} = -\left(\frac{\tau}{\kappa}\right)(s) t(s) - b(s) \tag{1.3}$$

along a timelike curve α under the condition that $\kappa(s) \neq 0$ and we call it modified Darboux vector field of a timelike curve α.

Definition 1.1. A timelike curve $\alpha : I \to \mathbb{R}^3_1$ with $\kappa(s) \neq 0$ is called a cylindrical helices if the tangent lines of α make a constant angle with a fixed direction.

It has been known that a timelike curve $\alpha(s)$ is a cylindrical helices if and only if $\left(\frac{\tau}{\kappa}\right)(s)$ is constant. We call a timelike curve circular helices if both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant [4].

Definition 1.2. Let α and $\overline{\alpha}$ be two regular curves with $\kappa(s) \neq 0$, $\bar{\kappa}(s) \neq 0$, $s \in I$. Let $\{t, n, b\}$ and $\{\overline{t}, \overline{n}, \overline{b}\}$ be the Frenet frame on $\mathbb{R}^3_1$ along α and $\overline{\alpha}$, respectively. If $\{n, \overline{n}\}$ is linearly dependent, the other words, if the principal normal lines of α and $\overline{\alpha}$ at $s \in I$ are equal, then the curve α is called a Bertrand curve. In this case $\overline{\alpha}$ is called a Bertrand mate of α and we can write

$$\overline{\alpha}(s) = \alpha(s) + rn(s), \forall s \in I. \tag{1.4}$$

The mate of Bertrand curve is denoted by $(\alpha, \overline{\alpha})$ [1].

Theorem 1.1. Let $(\alpha, \overline{\alpha})$ be a mate of Bertrand curve in Minkowski 3-space $\mathbb{R}^3_1$. Then r is constant which is defined as (1.4) [1].

Theorem 1.2. Let α and $\overline{\alpha}$ be two regular curves of Minkowski 3-space $\mathbb{R}^3_1$. Then $(\alpha, \overline{\alpha})$ is a mate of Bertrand curve if and only if there exists a linear relation

$$A\kappa + B\tau = 1, \tag{1.5}$$

where A, B are nonzero constants and κ and τ are curvature and torsion of α, respectively [1].

Corollary 1.1. Let α be a regular timelike curve of Minkowski 3-space $\mathbb{R}^3_1$. Then α has more than one Bertrand mate if and only if α is a circular helices.

Any plane curve α is a Bertrand curve whose Bertrand mates are parallel curves of α [1].

Corollary 1.2. Let $\alpha : I \to \mathbb{R}^3_1$ be a timelike curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then α is a Bertrand curve if and only if there exists a real number $A \neq 0$ such that

$$A(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)) - \tau'(s) = 0. \tag{1.6}$$

In this case the Bertrand mate of α is given by

$$\overline{\alpha}(s) = \alpha(s) + An(s).[8] \tag{1.7}$$

2. Curves on Timelike Ruled Surfaces

Let $\alpha : I \to \mathbb{R}^3_1$ be a differentiable timelike curve in Minkowski 3-space $\mathbb{R}^3_1$ parametrized by arc-length. The tangent vector field of a timelike curve α will be denoted by t.

A spacelike straight line,

$$\ell : \mathbb{R} \to \mathbb{R}^3_1,\ v \to \ell(v) = (\alpha_1(t) + va_1(t),\ \alpha_2(t) + va_2(t),\ \alpha_3(t) + va_3(t)),\ (2.1)$$

where the scalars $a_i(t) \in \mathbb{R}$ for all $1 \leq i \leq 3$, are the components of the director vector at the point $\alpha(t)$, can be chosen so that the director vector of ℓ and the tangent vector of a timelike curve α are linearly independent at every point of a timelike curve α.

As ℓ moves along a timelike curve α it generates a Ruled surface given by the parametrization $(I \times \mathbb{R}, \Psi)$, where

$$\Psi_{(\alpha,\ell)} : I \times \mathbb{R} \to \mathbb{R}^3_1 \qquad (2.2)$$

$$(t, v) \to \Psi_{(\alpha,\ell)}(t, v) = (\alpha_1(t) + va_1(t),\ \alpha_2(t) + va_2(t),\ \alpha_3(t) + va_3(t)),$$

which can be obtained in the Minkowski 3-space $\mathbb{R}^3_1$. We call α the base curve and ℓ the director curve. The straight lines $v \to \alpha(t) + v\ell(t)$ are called rulings [9].

In this section we study cylindrical helices and Bertrand curves from the view point of the theory of curves on timelike Ruled surfaces. Here, we say that a timelike Ruled surface $\Psi_{(\alpha,\ell)}$ is a developable surface if Gaussian curvature of the regular part of $\Psi_{(\alpha,\ell)}$ vanishes. By these facts, we now pay attention to Gaussian curvature and mean curvature of timelike Ruled surfaces. Let $\|\ell(t)\| = 1$. It is easy to show that Gaussian curvature of $\Psi_{(\alpha,\ell)}$ is

$$K(t, v) = -\frac{(\det(\ell^{'}(t), \alpha^{'}(t), \ell(t)))^2}{(EG - F^2)^2} \qquad (2.3)$$

and mean curvature of $\Psi_{(\alpha,\ell)}$ is

$$H(t, v) = \frac{\det(\alpha^{''}(t) + v\ell^{''}(t), \alpha^{'}(t) + v\ell^{'}(t), \ell(t)) - 2 < \alpha^{'}(t), \ell(t) > \det(\ell^{'}(t), \alpha^{'}(t), \ell(t))}{2(EG - F^2)^{\frac{3}{2}}}$$

where

$$E = E(t, v) = \left\|\alpha^{'}(t) + v\ell^{'}(t)\right\|^2,\ F = F(t, v) =< \alpha^{'}(t), \ell(t) >,\ G = G(t, v) = 1.$$

In particular Gaussian curvature of the rectifying developable of a timelike curve vanishes and mean curvature of the principal normal surface of a timelike curve is

$$H(s, v) = \frac{v(\tau^{'}(s) + v(\kappa(s)\tau^{'}(s) - \kappa^{'}(s)\tau(s))}{2(EG - F^2)^{\frac{3}{2}}} \qquad (2.4)$$

where s is the arc-length of a timelike curve α. It follows from this fact that $H(s, v) = 0$ if and only if $v = 0$ or $\tau^{'}(s) = v(\kappa^{'}(s)\tau(s) - \kappa(s)\tau^{'}(s))$. Thus, mean curvature of the principal normal surface $\Psi_{(\alpha,n)}$ of α always vanishes along a timelike curve α. If there exist a point $s_0 \in I$ such that $\kappa^{'}(s)\tau(s) - \kappa(s)\tau^{'}(s) = 0$, then $H(s_0, v_0) = 0$ for some $v_0 \neq 0$ if and only if $\tau^{'}(s_0) = 0$. In this case $\kappa^{'}(s_0) = 0$. Therefore, $H(s_0, v_0) = 0$ for some $v_0 \neq 0$ if and only if $\tau^{'}(s_0) = \kappa^{'}(s_0) = 0$ or

$$v_0 = \frac{\tau^{'}(s_0)}{\kappa^{'}(s)\tau(s) - \kappa(s)\tau^{'}(s)}. \tag{2.5}$$

If $\tau^{'}(s_0) \neq 0$ and $\kappa^{'}(s)\tau(s) - \kappa(s)\tau^{'}(s) = 0$, then $H(s_0, v) \neq 0$ for any $v \neq 0$. Moreover, under the assumption that $\tau^{'}(s_0) = \kappa^{'}(s_0) = 0$, $H(s_0, v) = 0$ for any v. Of course, if $\kappa^{'}(s)\tau(s) - \kappa(s)\tau^{'}(s) \neq 0$, mean curvature vanishes along the curve given by

$$\tilde{\alpha}(s) = \alpha(s) + \frac{\tau^{'}(s)}{\kappa^{'}(s)\tau(s) - \kappa(s)\tau^{'}(s)} n(s). \tag{2.6}$$

Let $\alpha : J \to \Psi_{(\alpha,\ell)}(I \times \mathbb{R}) \subset \mathbb{R}^3_1$ be a regular timelike curve. We say that a timelike curve α is the minimal locus of $\Psi_{(\alpha,\ell)}$ if mean curvature H of $\Psi_{(\alpha,\ell)}$ vanishes on $\alpha(J)$. By the above calculation and Corollary 1.2. , we have the following proposition.

Proposition 2.1. Let α be a Bertrand curve and $\bar{\alpha}$ be the Bertrand mate of α. Then $\bar{\alpha}$ is the minimal locus of the principal normal surface of α.

Proof. By Corollary 1.2., if $\bar{\alpha}$ is the Bertrand mate of α, then there exist a real number A such that $A(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)) - \tau'(s) = 0$ and $\bar{\alpha}(s) = \alpha(s) + An(s)$. This means that $H(\bar{\alpha}(s)) = H(s, A) = 0$. This completes the proof.

Proposition 2.2. Let $\Psi_{(\alpha,\ell)}(s, v) = \alpha(s) + v\ell(s)$ be a timelike Ruled surface with $\|\ell(s)\| = 1$. Let $\gamma(s) = \alpha(s) + v(s)\ell(s)$ be a curve on $\Psi_{(\alpha,\ell)}$, where s is the arc-length of $\gamma(s)$. Consider the following three conditions on γ :

(1) $\gamma(s)$ is a line of striction of $\Psi_{(\alpha,\ell)}$.

(2) $\gamma(s)$ is a geodesic of $\Psi_{(\alpha,\ell)}$.

(3) The angles between $\gamma^{'}(s)$ and $\alpha(s)$ are constant.

If assume that any two of the above three conditions hold, then the other condition holds.

We remark that the above conditions are respectively equivalent to the following conditions:

(1́) $<\gamma^{'}(s), \alpha^{'}(s)>= 0$.

(2́) $<\gamma^{''}(s), \alpha(s)>= 0$.

(3́) $<\gamma^{'}(s), \alpha(s)>=$constant.

The assertion follows from the fact that

$(<\gamma^{'}(s), \alpha(s)>)^{'} = <\gamma^{''}(s), \alpha(s)> + <\gamma^{'}(s), \alpha^{'}(s)>$.

Corollary 2.1. Suppose that there exist two disjoint geodesics $\gamma_i(s)$ $(i=1,2)$ on a timelike Ruled surface. $\Psi_{(\alpha,\ell)}(s,v)=\alpha(s)+v\ell(s)$ such that the angles between $\gamma_i^{'}(s)$ and $\ell(s)$ are constant. Then the timelike Ruled surface $\Psi_{(\alpha,\ell)}(s,v)$ is a cylindrical surface and both of $\gamma_i(s)$ are cylindrical helices. Moreover, the direction of $\ell(s)$ is equal to the direction of the Darboux vector of $\gamma_i(s)$.

Proof. By the above proposition, $\gamma_i(s)$ are lines of striction of $\Psi_{(\alpha,\ell)}$. If the point $\Psi_{(\alpha,\ell)}(s)$ is a non-cylindrical, then $\gamma_1(s)=\gamma_2(s)$ by the uniqueness of the line of striction, so that the timelike Ruled surface is a cylindrical surface. Since $\gamma_i(s)$ are geodesics of $\Psi_{(\alpha,\ell)}$, these are cylindrical helices and the rectifying plane of $\gamma_i(s)$ is the rectifying developable of $\gamma_i(s)$.

Corollary 2.1 gives a characterization of cylindrical surfaces by the existence of geodesics with special properties. Especially, a cylindrical surface is the rectifying developable of a cylindrical helices which is a geodesic of the original surface. We now consider the question when a Ruled surface is the rectifying developable of a curve.

Theorem 2.1. Let $\Psi_{(\alpha,\ell)}(s,v)=\alpha(s)+v\ell(s)$ be a non-singular timelike Ruled surface with $\|\ell(s)\|=1$. Let $\gamma(s)=\alpha(s)+v(s)\ell(s)$ be a curve on $\Psi_{(\alpha,\ell)}$ with $\kappa(s)\neq 0$, then the following conditions are equivalent:

(1) $\Psi_{(\alpha,\ell)}$ is the rectifying developable of $\gamma(s)$.

(2) $\gamma(s)$ is a geodesic of $\Psi_{(\alpha,\ell)}$ which is transversal to rulings and $\Psi_{(\alpha,\ell)}$ is a developable surface.

(3) $\gamma(s)$ is a geodesic of $\Psi_{(\alpha,\ell)}$ which is transversal to rulings and Gaussian curvature of $\Psi_{(\alpha,\ell)}$ vanishes along $\gamma(s)$.

Proof. Since the Darboux vector field always transverse to rulings, the condition (2) holds under the assumption of the condition (1). It is trivial that the condition (3) follows from the condition (2). We assume that the condition (3) holds. Since $\gamma(s)$ is transverse to rulings, we may assume that $\gamma(s)=\alpha(s)$. Gaussian curvature of $\Psi_{(\alpha,\ell)}$ is given by

$$K(s,v)=-\frac{(\det(\ell^{'}(s),\alpha^{'}(s),\ell(s)))^2}{(EG-F^2)^2} \tag{2.7}$$

then it vanishes along $\alpha(s)$ if and only if $\det(\ell^{'}(s),\alpha^{'}(s),\ell(s))=0$. Since $\alpha(s)$ is a geodesic of $\Psi_{(\alpha,\ell)}$, $\ell(s)$ is contained in the rectifying plane of α at $\alpha(s)$. There exist $\lambda(s)$, $\mu(s)$ such that $\ell(s)=\lambda(s)t(s)+\mu(s)b(s)$, where $t(s)=\alpha^{'}(s)$ and $b(s)$ is the binormal vector of α. By the Frenet-Serret Formulae, we have

$$\ell^{'}(s)=\lambda^{'}(s)t(s)+\mu^{'}(s)b(s)+(\lambda(s)\kappa(s)-\mu(s)t(s))n(s). \tag{2.8}$$

It follows from this formula that $\det(\ell^{'}(s),\alpha^{'}(s),\ell(s))=$ $(\mu(s)t(s)-\lambda(s)\kappa(s))\mu(s)$. If there exist a point s_0 such that $\mu(s_0)=0$, then $\ell(s_0)=\lambda(s_0)t(s_0)$. this contradicts to the assumption that α is transversal to rulings.

Hence, we have $\mu(s)t(s) - \lambda(s)\kappa(s) = 0$, so that

$$\tau(s)\ell(s) = \tau(s)\lambda(s)t(s) + \kappa(s)\lambda(s)b(s) = -\lambda(s)D(s) \tag{2.9}$$

where $D(s)$ is the Darboux vector field along α.

Since the rectifying developable of a cylindrical helices is a cylindrical surface, we have the following other characterization of cylindrical surfaces as a simple corollary of Theorem 2.1.

Corollary 2.2. Suppose that $\Psi_{(\alpha,\ell)}$ is a non-singular developable surface. If there exist a cylindrical helices with non-zero curvature on $\Psi_{(\alpha,\ell)}$ which is a geodesic of $\Psi_{(\alpha,\ell)}$ then $\Psi_{(\alpha,\ell)}$ is a cylindrical surface.

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