

ALMOST CONVERGENCE AND SOME MATRIX TRANSFORMATIONS

Qamaruddin and S. A. Mohiuddine

Department of Mathematics, Aligarh Muslim University
Aligarh-202002, India
sdqamar@rediffmail.com, mohiuddine@math.com

Abstract

In this paper we characterize the matrix classes $(l(p, u), f_\infty)$ and $(l(p, u), f)$ which generalize the matrix classes given by Mursaleen [6].

Mathematics Subject Classification: 40C05, 40H05

Keywords: Banach limit, almost convergence, matrix transformation.

1. Introduction

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ respectively with the usual norm $\|x\| = \sup_k |x_k|$. A continuous linear functional ϕ on l_∞ is called a Banach limit [1] if (i) $\phi(x) \geq 0$ for $x = (x_k)$, $x_k \geq 0$ for every k , (ii) $\phi(x_{k+1}) = \phi(x_k)$, and (iii) $\phi(e) = 1$ where $e = (1, 1, 1, \dots)$. A sequence $x \in l_\infty$ is said to be almost convergent to the value L if all of its Banach limits equal to L (see [3]). We denote the set of all almost convergent sequences by f , i.e.

$$f := \{x \in l_\infty : \lim_m t_{mn}(x) = L, \text{ uniformly in } n\},$$

where

$$t_{mn}(x) = \frac{1}{m+1} \sum_{k=0}^m x_{k+n}, \quad t_{-1,n} = 0,$$

and

$$L = f - \lim x$$

Nanda [7] has defined a new set of sequences as follows

$$f_\infty := \{x \in l_\infty : \sup |t_{mn}(x)| < \infty\},$$

we call f_∞ the set of all almost bounded sequences.

Bullet and Cakar [2] have defined a sequence space $l(p, s)$ and characterized the matrix classes $(l(p, s), l_\infty)$ and $(l(p, s), c)$.

In this paper we characterize the matrix classes $(l(p, s), l_\infty)$ and $(l(p, s), f)$, where $l(p, u)$ is more general than $l(p, s)$.

Let $p = (p_k)$ be a sequence of real numbers with $p_k > 0$. The space $l(p, s)$ is defined as (see [2])

$$l(p, s) := \left\{ x : \sum_k k^{-s} |x_k|^{p_k} < \infty \right\}$$

Let $u = (u_k)$ be a sequence of real numbers such that $u_k \neq 0$ ($k = 1, 2, \dots$) and $u^{-1} = (u_k^{-1})$.

The space $l(p, u)$ is defined as (see [5])

$$l(p, u) := \left\{ x : \sum_k |u_k x_k|^{p_k} < \infty \right\}.$$

If we take $u = (u_k)$ defined by

$$u_k = k^{-s/p_k} \quad , \quad s \geq 0 \quad , \quad k = 1, 2, \dots$$

then $l(p, u)$ reduces to $l(p, s)$. Obviously $x \in l(p, u)$ has same meaning as $u \cdot x \in l(p)$, and so $l(p, u)$ is paranormed by

$$g(x) = \left[\sum_k |u_k x_k|^{p_k} \right]^{1/M} \quad , \quad M = \max(1, \sup p_k).$$

Let X and Y be two sequence spaces and $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n . If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y . By (X, Y) we denote the class of matrices A such that $Ax \in Y$ for $x \in X$.

2. MAIN RESULTS

Throughout the text, we use the following notation :

For all integers $n, m \geq 1$,

$$\begin{aligned} t_{mn}(Ax) &= \frac{1}{m+1} \sum_{i=0}^m A_{n+i}(x) \\ &= \sum_k a(n, k, m) x_k \end{aligned}$$

where

$$a(n, k, m) = \frac{1}{m+1} \sum_{i=0}^m a_{n+i, k}$$

Theorem 2.1. Let $1 < p_k < \sup_k p_k = H < \infty$ for every k . Then $A \in (l(p, u), f_\infty)$ if and only if there exists an integer $N > 1$ such that

$$(2.1.1) \quad \sup_{m, n} \sum_k |a(n, k, m)|^{q_k} u_k^{-q_k} N^{-q_k} < \infty.$$

Proof. Sufficiency. Let (2.1.1) hold and that $x \in l(p, u)$ using the following inequality (see [4])

$$|ab| \leq C(|a|^q C^{-q} + |b|^p)$$

for $C > 0$ and a, b two complex numbers, we have

$$\begin{aligned} t_{mn}(Ax) &= \sum_k |a(n, k, m) u_k^{-1} u_k x_k| \\ &\leq \sum_k N [|a(n, k, m)|^{q_k} u_k^{-q_k} N^{-q_k} + |u_k x_k|^{p_k}] \end{aligned}$$

Taking the supremum over m, n on both sides and using (2.1.1), we get $Ax \in f_\infty$ for $x \in l(p, u)$, i.e. $A \in (l(p, u), f_\infty)$.

Necessity. Let $A \in (l(p, u), f_\infty)$. Write $q_n(x) = \sup_m |t_{mn}(Ax)|$. It is easy to see that for $n \geq 0$, q_n is a continuous seminorm on $l(p, u)$ and (q_n) is pointwise bounded on $l(p, u)$. Suppose that (2.1.1) is not true. Then there exists $x \in l(p, u)$ with $\sup_n q_n(x) = \infty$. By the principle of condensation of singularities [8], the set

$$\{x \in l(p, u) : \sup_n q_n(x) = \infty\}$$

is of second category in $l(p, u)$ and hence nonempty, that is, there is $x \in l(p, u)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that (q_n) is pointwise bounded on $l(p, u)$. Now by the Banach-Steinhaus theorem, there is constant M such that

$$(2.1.2) \quad q_n(x) \leq Mg(x).$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \delta^{M/p_k} (\operatorname{sgn} a(n, k, m)) |a(n, k, m)|^{q_k-1} u_k^{-1} S^{-1} N^{-q_k/p_k}, & 1 \leq k \leq k_0 \\ 0 & \text{for } k > k_0; \end{cases}$$

where $0 < \delta < 1$ and

$$S = \sum_{k=1}^{k_0} |a(n, k, m)|^{q_k} N^{-q_k}.$$

Then it is easy to see that $x \in l(p, u)$ and $g(x) \leq \delta$. Applying this sequence to (2.1.2) we get the condition (2.1.1).

Theorem 2.2. Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every k . Then

$A \in (l(p, u), f)$ if and only if

- (i) the condition (2.1.1) of Theorem 2.1 holds,
- (ii) $\lim_m a(n, k, m) = \alpha_k$ uniformly in n , for every k .

Proof. Sufficiency. Let (i) and (ii) hold and $x \in l(p, u)$.

For $j \geq 1$

$$\begin{aligned} \sum_{k=1}^j |a(n, k, m)|^{q_k} N^{-q_k} u_k^{-q_k} \\ \leq \sup_m \sum_k |a(n, k, m)|^{q_k} N^{-q_k} u_k^{-q_k} < \infty \quad \text{for every } n. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_k |\alpha_k|^{q_k} N^{-q_k} u_k^{-q_k} &= \lim_j \lim_m \sum_{k=1}^j |a(n, k, m)|^{q_k} N^{-q_k} u_k^{-q_k} \\ &\leq \sup_m \sum_k |a(n, k, m)|^{q_k} N^{-q_k} u_k^{-q_k} < \infty. \end{aligned}$$

Consequently the series $\sum_k a(n, k, m)x_k$ and $\sum_k \alpha_k x_k$ converge for every n, m ; and for every $x \in l(u, p)$. Now, for $\epsilon > 0$ and $x \in l(u, p)$, choose $k_0 \geq 1$ such that

$$\sum_{k \geq k_0+1} |u_k^{-1} x_k|^{p_k} \leq 1$$

By(ii), there exists m_0 such that

$$\left| \sum_{k=1}^{k_0} [a(n, k, m) - \alpha_k] \right| < \infty$$

for every $m > m_0$. By condition (i), it follows that

$$\left| \sum_{k \geq k_0+1} [a(n, k, m) - \alpha_k] \right|$$

is arbitrarily small. Therefore

$$(*) \quad \lim_m \sum_k a(n, k, m)x_k = \sum_k \alpha_k x_k$$

uniformly in n . Hence $A \in (l(p, u), f)$.

Necessity. Let $A \in (l(p, u), f)$. Since $f \subset f_\infty$ (see [6]), condition (i) follows by Theorem 2.1. Since $e_k \in l(p, u)$, condition (ii) follows immediately by (*).

References

- [1] S. Banach, *Theòrie des operations linéaires*, Warsaw, 1932.
- [2] E. Bullut, and O, Çakar, The sequence space $l(p, s)$ and related matrix transformations, *Comm. Fac. Sci. Univ. Aukara (A₁)*, 28 (1979), 33-44.
- [3] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Math.*, 80 (1948), 167-190.
- [4] I. J. Maddox, Continuous and Kóthe-Toeplitz dual of certain sequence spaces, *Proc. Camb. Phil. Soc.*, 65 (1969), 413-435.
- [5] Mursaleen and A. A. Khan, The sequence space $l(p, s)$ and some matrix transformation, *J. Fac. Education*, 1 (1) (1994), 9-13.
- [6] Mursaleen, Infinite matrices and almost convergent sequences, *South-east Asian Bull. Math.*, 19 (1) (1995), 45-48.
- [7] S. Nanda, On some sequence spaces, *Math. Student*, 48 (4) (1980), 348-352.
- [8] K. Yosida, *Funtional Analysis*, Springer-Verlag, Berlin Heidelberg, New York, 1966.

Received: September 10, 2005