

A Note on Certain Stability and Limiting Properties of ν -infinitely divisible distributions

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Abstract. The class of ν -infinitely divisible (ID) distributions, which arise in connection with random summation, is a reach family including geometric infinitely divisible (GID) and geometric stable (GS) laws. We present two simple results connected with triangular arrays with random number of terms and their limiting ν -ID distributions as well as random sums with ν -ID distributed terms. These generalize and unify certain results scattered in the literature that concern the special cases of GID and GS laws.

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1. INTRODUCTION AND THE MAIN RESULTS

Let X be an infinitely divisible (ID) random variable (RV), so that for each integer $n \geq 1$ its characteristic function (ChF) ϕ admits the representation $\phi(t) = [\phi_n(t)]^n$, where $\phi_n = \phi^{1/n}$ is another ChF corresponding to some RV $X^{(n)}$. Then, the sum $X_1^{(n)} + \dots + X_n^{(n)}$ of n independent and identically distributed (IID) copies of $X^{(n)}$ converges to X (the sum actually has the same distribution as X). It now follows from transfer theorems (see, e.g., [4], Theorem 4.1.2) that the *random sums* $X_1^{(n)} + \dots + X_{\nu_n}^{(n)}$, where (ν_n) is a sequence of integer-valued RV's such that $\nu_n \xrightarrow{p} \infty$ (in probability) while $\nu_n/n \xrightarrow{d} \nu$ (in distribution), converge in distribution to a RV Y whose ChF is of the form

$$(1) \quad \psi(t) = \lambda(-\log \phi(t)),$$

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where λ is the Laplace transform (LT) of the RV ν (If the variables X and Y are non-negative with LT's ϕ and ψ , respectively, then the same relation holds, see, e.g., Remark 4.1.1 in [4]. Consequently, for simplicity we shall follow the convention that these are LT's whenever the relevant variables are non-negative).

The following is a slight generalization, which unifies many results concerning sums with geometric number of terms, scattered in the literature (see examples below).

Proposition 1.1. *Assume that zero is an accumulation point of $\Delta \subset (0, 1)$, and consider a family $\{\nu_p, p \in \Delta\}$ of integer-valued, nonnegative RV's such that $p\nu_p \xrightarrow{d} \nu$ as $p \rightarrow 0$. Further, for each $p \in \Delta$ let $X_1^{(p)}, X_2^{(p)}, \dots$ be a sequence of IID RV's that are independent of ν_p and given by the ChF $\phi_p(t) = \phi^p(t)$, where ϕ is an ID ChF. Then, as $p \rightarrow 0$, the random sums*

$$(2) \quad S_p = X_1^{(p)} + \dots + X_{\nu_p}^{(p)}$$

converge in distribution to a RV Y whose ChF is given by (1), where λ is the LT of ν . [If the above RV's are non-negative then the same relation applies to their LT's.]

Proof. For simplicity, we shall assume that the RV's are nonnegative and work with the relevant LT's (the general case involving the ChF's is identical). Writing the LT of the random sum S_p as

$$\begin{aligned} \mathbb{E}e^{-tS_p} &= \sum_n [\mathbb{E}e^{-tX_1^{(p)}}]^n \mathbb{P}(\nu_p = n) = \mathbb{E}[\phi^p(t)]^{\nu_p} \\ &= \mathbb{E}[\phi(t)]^{p\nu_p} = \mathbb{E}e^{p\nu_p \log \phi(t)} = \lambda_p(-\log \phi(t)), \end{aligned}$$

where λ_p is the LT of $p\nu_p$, we conclude that it must converge to (1) since $p\nu_p$ converges in distribution to ν and λ is the LT of ν . \square

One important special case arises when ν_p above has a geometric distribution with parameter $p \in \Delta = (0, 1)$, so that

$$(3) \quad \mathbb{P}(\nu_p = k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \dots$$

In this case $p\nu_p$ converges in distribution to the standard exponential variable with the LT $\lambda(t) = (1+t)^{-1}$, and the ChF (or LT for positive variables) of the limiting distribution of the random sums S_p is given by

$$(4) \quad \psi(t) = \frac{1}{1 - \log \phi(t)}.$$

Since their introduction in [9], the distributions given by (4) are known as *geometrically infinitely divisible* laws, as they can be decomposed into geometric convolutions. More precisely, if Y is a RV with the ChF (or LT) (4) above,

then for any $p \in \Delta$ there exists a sequence $(X_j^{(p)})$ of IID RV's independent of ν_p such that we actually have the equality in distribution,

$$(5) \quad Y \stackrel{d}{=} \sum_{j=1}^{\nu_p} X_j^{(p)}.$$

It is worth noting that not all distributions with the ChF (1) admit the random divisibility property (5). As shown in [4, 10], those that do must have a special structure: the probability generating functions (PGF's) generated by the family $\{\nu_p, p \in \Delta\}$ must form a *commutative* semigroup with the operation of superposition. It can be verified easily that this is the case when ν_p is geometric (3):

$$G_p \circ G_q(z) = G_p(G_q(z)) = G_q(G_p(z)) = G_q \circ G_p(z), \quad p, q \in \Delta = (0, 1),$$

where

$$(6) \quad G_p(z) = \mathbb{E}z^{\nu_p} = \frac{pz}{1 - (1-p)z}$$

is the PGF of ν_p . More general distributions with this property are known as ν -infinitely divisible laws. Their ChF's (or LT's) are of the form (1) where ϕ is ID and the LT λ is connected with the family $\{\nu_p, p \in \Delta\}$ via the relation

$$(7) \quad G_p(z) = \lambda \left(\frac{1}{p} \lambda^{-1}(z) \right), \quad p \in \Delta, 0 < z \leq 1,$$

where G_p is the PGF of ν_p (see [4, 10] for details). The following result holds in the above setting.

Proposition 1.2. *Consider a family $\{\nu_p, p \in \Delta\}$ of integer-valued, nonnegative RV's such that $\mathbb{E}\nu_p = 1/p \rightarrow \infty$ and the corresponding PGF's G_p form a commutative semigroup under the operation of composition. Let Y_1, Y_2, \dots be IID variables independent of the ν_p 's, with the ChF ψ given by (1), where ϕ is a ID ChF and λ satisfies (7). Then the ChF of the random sum*

$$(8) \quad Y_p = Y_1 + \dots + Y_{\nu_p}$$

is also given by (1), but with ϕ replaced by $\phi^{1/p}$. [If the above RV's are non-negative then the ChF's in these relations can be replaced by the LT's.]

Proof. For simplicity, we shall assume that the RV's are nonnegative and work with the relevant LT's (the general case involving the ChF's is identical). Since the Y_i 's are IID with the LT ψ given by (1) with λ satisfying (7), the LT of Y_p is

$$\begin{aligned} \mathbb{E}e^{-tY_p} &= G_p(\psi(t)) = G_p(\lambda(-\log \phi(t))) \\ &= \lambda \left(\frac{1}{p} \lambda^{-1}(\lambda(-\log \phi(t))) \right) = \lambda(-\log \phi^{1/p}(t)), \end{aligned}$$

and the result follows. □

Remark 1. Note that if the variables $p\nu_p$ in Proposition 1.1 converge to 1, then $\psi(t) = \phi(t)$. This is, for example, the case when ν_p has the Poisson distribution with mean $1/p$, where $p \in \Delta = (0, 1)$. However, the PGF's corresponding to this family do not generate commutative semigroup, so that Proposition 1.2 is not applicable in this case.

Remark 2. If ϕ in (1) corresponds to a stable law (see, e.g., [21]), then the distributions corresponding to ψ are called ν -stable laws (see, e.g., [12, 13]). Suppose further that ϕ corresponds to a *strictly stable* distribution with index $\alpha \in (0, 2]$, so that for all $c > 0$ and $t \in \mathbb{R}$ we have $\phi^c(t) = \phi(c^{1/\alpha}t)$. Then, if the PGF's of the family $\{\nu_p, p \in \Delta\}$ and the LT of the limit of $p\nu_p$ are connected via (7), Proposition 1.2 shows that for each $p \in \Delta$, the random sum (8) of the Y_i 's has the same *type* of distribution as each of the terms in the sum. Such *random stability* properties were studied in [2, 8, 22], among others. Some specific examples will be considered below.

Example 1. Consider a trivial case when the distribution of X is concentrated at 1, so that its LT is $\phi(t) = e^{-t}$. Then by Proposition 1.1 the LT of the limiting distribution of the random sums (2) is $\psi(t) = \lambda(s)$. It is not surprising that this coincides with the LT of ν , which is the limit of $p\nu_p$, since here $S_p = p\nu_p$. Further, assume that ν_p is geometric (3) so that the corresponding PGF's form a commutative semigroup and the limit ν of $p\nu_p$ as $p \rightarrow 0$ is a standard exponential variable with the LT $\lambda(t) = (1+t)^{-1}$. Then Proposition 1.2 implies that for each $p \in (0, 1)$ the LT of the random sum (8) is $(1 + t/p)^{-1}$, which is again an exponential variable. Thus we recover the well-known stability property of the exponential distribution with respect to geometric summation (see, e.g., [1]).

Example 2. Let $\phi(t) = e^{-t^2}$ be the ChF of a normal distribution with mean zero and variance 2. Then by Proposition 1.1 the ChF of the limiting distribution of the random sums (2) with geometrically distributed ν_p is $\psi(t) = (1+t^2)^{-1}$ - the standard classical Laplace distribution (see, e.g., [11]). Proposition 1.2 now implies that for each $p \in (0, 1)$ the ChF of the random sum (8) is $(1 + t^2/p)^{-1}$, which is again a Laplace ChF. We thus obtained the stability property of the Laplace distribution with respect to geometric summation (see, e.g., [11]).

Example 3. Consider now the LT $\phi(t) = e^{-t^\alpha}$, $\alpha \in (0, 1)$, corresponding to a positive stable law, and let the ν_p 's again be geometric, so that ν is standard exponential as before. Then, in view of Proposition 1.1 we obtain the LT of the limiting distribution of (2) to be $\psi(t) = (1+t^\alpha)^{-1}$. This is the standard Mittag-Leffler distribution introduced in [19]. The application of Proposition 1.2 shows that we have the stability property with respect to geometric summation as well (as expected, since ϕ is strictly stable).

Example 4. Now take $\phi(t) = e^{-|t|^\alpha}$, $\alpha \in (0, 2)$, to be the ChF of a symmetric stable law with index α (which reduces to the normal when $\alpha = 2$), and let ν_p be

geometric as before. Again, by Proposition 1.1 we obtain $\psi(t) = (1 + |t|^\alpha)^{-1}$, which corresponds to the *Linnik distribution* (see, e.g., [11] and references therein), also known as α -Laplace law (see [18]). The application of Proposition 1.2 again recovers the well-known stability property of this distribution with respect to geometric summation (see, e.g., [15]).

Let us note that the distributions that arise in these four examples are all special cases of geometric stable (GS) laws (see, e.g., [14]), which are given by the ChF $\psi(t) = (1 - \log \phi(t))^{-1}$ with a stable ChF ϕ . These infinitely divisible distributions appear as weak limits of (normalized) sums of independent and identically distributed (IID) RV's, where the number of terms in the summation has a geometric distribution (independent of the terms) with the mean converging to infinity. More information on theory and applications of these classes of distributions can be found in [8].

Below we present additional examples where this time ϕ corresponds to the above GS distributions and ν_p are assumed to be geometrically distributed.

Example 5. If $\phi(t) = (1+t)^{-1}$ is the LT of a standard exponential distribution, then by Proposition 1.1 we obtain $\psi(t) = (1 + \log(1+t))^{-1}$, which is the geometric exponential distribution introduced in [20] and studied in [7]. We thus recovered Theorem 3.1 of [20] (as well as Theorem 2.1 of [7]). Further, Proposition 1.2 shows that for each $p \in (0, 1)$ the LT of the random sum (8) is $(1 + (1/p) \log(1+t))^{-1}$, which corresponds to geometric gamma distribution (see [7]). Note that in this context Proposition 1.2 is a generalization of Lemma 3.2 of [20]. Similarly, if $\phi(t) = (1+t)^{-v}$, $v > 0$, is the LT of a standard gamma distribution, then $\psi(t) = (1 + v \log(1+t))^{-1}$ corresponds to the geometric gamma distribution, which recovers Theorem 2.2 of [7].

Example 6. If $\phi(t) = (1+t^\alpha)^{-1}$ is the LT of the standard Mittag-Leffler distribution, then by Proposition 1.1 we obtain $\psi(t) = (1 + \log(1+t^\alpha))^{-1}$, which is the geometric Mittag-Leffler distribution discussed in [6]. This recovers Theorem 2.2 of [6]. In turn, Proposition 1.2 shows that for each $p \in (0, 1)$ the LT of the random sum (8) is $(1 + (1/p) \log(1+t^\alpha))^{-1}$, which corresponds to geometric quasi factorial gamma distribution in the terminology of [6]. Moreover, this recovers Theorem 2.1 of [6]. Similarly, if $\phi(t) = (1+t^\alpha)^{-v}$, $v > 0$, is the LT of a generalized Mittag-Leffler distribution (or quasi factorial gamma distribution in the terminology of [6]), then $\psi(t) = (1 + v \log(1+t^\alpha))^{-1}$ is the geometric quasi factorial gamma distribution and we recover Theorem 2.3 of [6].

Example 7. Let $\phi(t) = (1+t^2)^{-1}$ be the ChF of the standard Laplace distribution, in which case ϕ^p corresponds to another ID distribution called Bessel function distribution (see [11], Chapter 4) or generalized Laplace distribution (see [16]). Then by Proposition 1.1 we obtain $\psi(t) = (1 + \log(1+t^2))^{-1}$, which is the geometric Laplace distribution defined in [23]. This recovers Theorem 1 of [23]. Similarly, Proposition 1.2 shows that for each $p \in (0, 1)$ the ChF of (8) is $(1 + (1/p) \log(1+t^2))^{-1}$.

Example 8. Let $\phi(t) = (1 + |t|^\alpha)^{-1}$, $0 < \alpha < 2$, be the ChF of the standard Linnik distribution, in which case ϕ^p corresponds a generalized Linnik distribution (see, e.g., [3, 17]), also known as generalized α -Laplace variable (see [23]). Again, by Proposition 1.1, we obtain $\psi(t) = (1 + \log(1 + |t|^\alpha))^{-1}$, which is the geometric α -Laplace distribution defined in [23]. This recovers Theorem 3 of [23]. Similarly, Proposition 1.2 shows that for each $p \in (0, 1)$ the ChF of (8) is $(1 + (1/p) \log(1 + |t|^\alpha))^{-1}$.

In closing, let us mention two other families $\{\nu_p, p \in \Delta\}$ related to the geometric distribution, which lead to further examples and special cases of our results. The first one is the *negative binomial* family given by the PGF

$$(9) \quad G_p(z) = \left(\frac{p}{1 - (1-p)z} \right)^r, \quad p \in \Delta = (0, 1), r > 0.$$

Here, the variables $p\nu_p$ converge in distribution to a standard gamma variable ν with shape parameter r (and scale 1), so that $\lambda(t) = (1+t)^{-r}$. Consequently Proposition 1.1 leads to further examples of ID distributions, in parallel to those “generated” by the geometric family and stable or geometric stable ChF’s (or LT’s) ϕ . However, this family of PGF’s is not commutative, so Proposition 1.2 does not apply. On the other hand, it is easy to see that whenever PGF’s G_p and LT λ (corresponding to ID laws) are connected via (7) then so are $G_p^{(k)}$ and LT λ_k for each integer $k \geq 1$, defined as

$$(10) \quad G_p^{(k)}(z) = [G_p(z^k)]^{1/k}, \quad \lambda_k(t) = [\lambda(kt)]^{1/k}.$$

Thus, another group of results and additional examples can be generated via Proposition 1.2 (as well as Proposition 1.1) when we take the $G_p^{(k)}$ ’s corresponding to the geometric laws (6). Note that this PGF’s correspond to the RV’s $1 + kN_p$, where N_p has the negative binomial distribution (9) with parameter $r = 1/k$ (so that the corresponding values are $1, 1 + k, 1 + 2k, \dots$). This distribution appears in [5], and was mentioned in [2, 22] in connection with random stability.

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