On a Class of Telescopic Numerical Semigroups

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Abstract

In this paper, we give some results on the class $S = \langle a, a + 2, 2a + 1 \rangle$ telescopic numerical semigroups where $a > 2$ and $a$ is even integer number.

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1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots n, \cdots\}$ and $S \subseteq \mathbb{N}$. $S$ is called a numerical semigroup if $S$ is sub-semigroup of $(\mathbb{N}, +)$ with $0 \in S$. It is known that every numerical semigroup is finitely generated, i.e. there exist elements of $S$, say $n_0, n_1, \cdots, n_p$ such that $n_0 < n_1 < \cdots < n_p$ and

$$S = \langle n_0, n_1, \cdots, n_p \rangle = \{ \sum_{i=0}^{p} k_in_i : k_i \in \mathbb{N} \}$$

and

$$G.C.D.(n_0, n_1, \cdots, n_p) = 1 \iff \text{Card}(\mathbb{N}\backslash S) < \infty$$

by [1]. For $S$ numerical semigroup, we define the following:

$$g(S) = max\{x \in \mathbb{Z} : x \notin S\}$$

is called the Frobenius number of $S$, where $\mathbb{Z}$ is the integer set. Thus, $S$ numerical semigroup is $S = \{0, n_0, n_1, \cdots, g(S) + 1 \rightarrow \cdots\}$ ( The arrow "$\rightarrow"$ means that every integer which is greater then $g(S) + 1$ belongs to $S$).

We say that a numerical semigroup is symmetric if for every $x \in \mathbb{Z}\backslash S$, we have $g(S) - x \in S$. by [2] For $n \in S \backslash \{0\}$, we define the Apéry set of the element $n$ as the set
semigroup, where \( a > 1 \). For \( S \) is symmetric and the corner of \( Ap \) by [3]. Finally, numerical semigroup \( Ap \) is called the corner of telescopic semigroup if \( s \) is even integer number.

It can easily be proved that \( Ap(S, n) \) is formed by the smallest elements of \( S \) belonging to the different congruence classes mod\( n \). Thus, \( \sharp(\text{Ap}(S, n)) = n \) and \( g(S) = \text{max}(\text{Ap}(S, n)) - n \), where \( \sharp(A) \) stands for cardinality(\( A \)) by [2]. The elements of \( \mathbb{N}\backslash S \), denoted by \( H(S) \), are called as gaps of \( S \). A gap \( x \) of a numerical semigroup \( S \) is fundamental if \( \{2x, 3x\} \subseteq S \). We denote by \( FH(S) \) the set of fundamental gaps of \( S \) by [4]. Let \( S = \langle s_1, s_2, \cdots, s_n \rangle \) be a numerical semigroup where \( k \in \{1, 2, \cdots, n - 1\} \) and \( s_k \leq u \). Then, \( u \in \langle s_1, s_2, \cdots, s_{n-1} \rangle \) is called the corner of \( \text{Ap}(S, s_n) \) if \( u \notin \text{Ap}(S, s_n) \) and \( (u - s_i) \in \text{Ap}(S, s_n) \) by [3]. Finally, numerical semigroup \( S = \langle s_1, s_2, s_3 \rangle \) is called as triply-generated telescopic semigroup if \( s_3 \in \langle \frac{s_1}{d}, \frac{s_2}{d} \rangle \), where \( d = g.c.d\{s_1, s_2\} \) by [5].

2. Main results

In this section, we give some results for numerical semigroup \( S = \langle a, a + 2, 2a + 1 \rangle \), where \( a > 2 \) and \( a \) is even integer number.

**Theorem 1:** Let \( S = \langle a, a + 2, 2a + 1 \rangle \). Then, \( S \) is a telescopic numerical semigroup, where \( a > 2 \) and \( a \) is even integer number.

**Proof:** Since \( a > 2 \) and \( a \) is even integer numbers, \( d = g.c.d\{a, a + 2\} = 2 \). Therefore, because of \( 2a + 1 = \frac{4a}{2} + \frac{a}{2} - \frac{a}{2} + 1 = 3\left(\frac{a}{2}\right) + 1\frac{a+2}{2} \), we obtain \( (2a + 1) \in \langle \frac{2}{2}, \frac{a+2}{2} \rangle \).

**Theorem 2:** The Frobenius number of the telescopic numerical semigroup \( S = \langle a, a + 2, 2a + 1 \rangle \) is \( g(S) = \frac{a^2}{2} + a - 1 \) and \( g(S) \) is odd, where \( a > 2 \) and \( a \) is even integer number.

**Proof:** Putting \( a_1 = a, a_2 = a + 2 \) and \( d = 2 \) from [5, proposition 2.2], we write \( g(S)) = \frac{a^2}{2} + a - 1 \). On the other hand, if we get \( a = 2k, k > 1 \), we obtain that \( g(S) = 2k^2 + 2k - 1 \) and \( g(S) \) is odd.

**Theorem 3:** Let \( S = \langle a, a + 2, 2a + 1 \rangle \). Telescopic numerical semigroup \( S \) is symmetric and the corner of \( \text{Ap}(S, 2a + 1) \) is \( 4a + 2 \), where \( a > 2 \) and \( a \) is even integer number.

**Proof:** It is clear that telescopic numerical semigroup \( S \) is symmetric from [5]. For \( S = \langle a, a + 2, 2a + 1 \rangle \), and \( u \in \langle a, a + 2 \rangle \), we will show that the number which satisfies the following conditions is \( u = 4a + 2 \):

(i) \( u \notin \text{Ap}(S, 2a + 1) \) since \( (4a + 2) - (2a + 1) = (2a + 1) \in S \) and \( (2a + 1) - (2a + 1) = 0 \in S \).

(ii) \( (u - a) \in \text{Ap}(S, 2a + 1) \) since \( (4a + 2) - (a) = (3a + 2) \in S \) and \( (3a + 2) - (2a + 1) = (a + 1) \notin S \).
(iii) \((u - (a + 2)) \in Ap(S, 2a + 1)\) since \((4a + 2) - (a + 2) = (3a) \in S\) and 
\((3a) - (2a + 1) = (a - 1) \notin S\).

**Example:** Let \(a = 4\). In this case, we write \(S = \langle 4, 6, 9 \rangle = \{0, 4, 6, 8, 9, 10, 12, \ldots \}\). \(S\) is telescopic numerical semigroup since \(d = g.c.d\{4, 6\} = 2\) and 
\(9 \in \langle \frac{4}{2}, \frac{6}{2} \rangle = \langle 2, 3 \rangle\). On the other hand, we obtain \(g(S)) = \frac{a^2}{2} + a - 1 = 11\), and 
the corner of \(Ap(S, 9) = \{s \in S : (s - 9) \notin S\} = \{0, 4, 6, 8, 10, 12, 16, 20\}\) is 18. 
Finally, we find that \(H(S) = \{1, 2, 3, 5, 7, 11\}\) and \(FH(S) = \{2, 3, 5, 7, 11\}\).

**References**


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