Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 2, 81 - 83

On a Class of Telescopic Numerical Semigroups

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Abstract

In this paper, we give some results on the class $S = \langle a, a+2, 2a+1\rangle$ telescopic numerical semigroups where $a > 2$ and a is even integer number.

Mathematics Subject Classification: 20M14, 20F50

Keywords: Numerical semigroups, telescopic, gaps, Apéry set.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots n, \cdots\}$ and $S \subseteq \mathbb{N}$. S is called a numerical semigroup if S is sub-semigroup of $(\mathbb{N}, +)$ with $0 \in \ S$. It is known that every numerical semigroup is finitely generated, i.e. there exist elements of S, say $n_0, n_1, \cdots, n_p$ such that $n_0 < n_1 < \cdots < n_p$ and

$$S = \langle n_0, n_1, \cdots, n_p\rangle = \{\sum_{i=0}^{p} k_i n_i \quad : k_i \in \mathbb{N}\}$$

and

$$G.C.D.(n_0, n_1, \cdots, n_p) = 1 \Leftrightarrow Card(\mathbb{N}\backslash S) < \infty$$

by [1]. For S numerical semigroup, we define the following:

$g(S) = max\{x \in \mathbb{Z} : x \notin \ S\}$ is called the Frobenius number of S, where $\mathbb{Z}$ is the integer set. Thus, S numerical semigroup is $S = \{0, n_0, n_1, \cdots, g(S) + 1 \rightarrow \cdots\}$ (The arrow "$\rightarrow$" means that every integer which is greater then $g(S) + 1$ belongs to S).

We say that a numerical semigroup is symmetric if for every $x \in \mathbb{Z}\backslash S$, we have $g(S) - x \in S$. by [2] For $n \in S\backslash\{0\}$, we define the Apéry set of the element n as the set

$$Ap(S,n) = \{s \in S : s - n \notin S\}.$$

It can easily be proved that $Ap(S,n)$ is formed by the smallest elements of S belonging to the different congruence classes $modn$. Thus, $\sharp(Ap(S,n)) = n$ and $g(S) = max(Ap(S,n)) - n$, where $\sharp(A)$ stands for $cardinality(A)$ by [2].
The elements of $\mathbb{N}\backslash S$, denoted by $H(S)$,are called as gaps of S.A gap x of a numerical semigroup S is fundamental if $\{2x, 3x\} \subset S$. We denote by $FH(S)$ the set of fundamental gaps of S by [4].Let $S = \langle s_1, s_2, \cdots, s_n\rangle$ be a numerical semigroup where $k \in \{1, 2, \cdots n-1\}$ and $s_k \leq u$. Then, $u \in \langle s_1, s_2, \cdots, s_{n-1}\rangle$ is called the corner of $Ap(S, s_n)$ if $u \notin\ Ap(S, s_n)$ and $(u - s_i) \in Ap(S, s_n)$ by [3].Finally,numerical semigroup $S = \langle s_1, s_2, s_3\rangle$ is called as triply-generated telescopic semigroup if $s_3 \in \langle \frac{s_1}{d}, \frac{s_2}{d}\rangle$, where $d = g.c.d\{s_1, s_2\}$ by [5].

2. Main results

In this section,we give some results for numerical semigroup $S = \langle a, a+2, 2a+1\rangle$, where $a > 2$ and a is even integer number.

Theorem 1: Let $S = \langle a, a+2, 2a+1\rangle$. Then,$S$ is a telescopic numerical semigroup,where $a > 2$ and a is even integer number.

Proof: Since $a > 2$ and a is even integer numbers,$d = g.c.d\{a, a+2\} = 2$. Therefore,because of $2a + 1 = \frac{4a}{2} + \frac{a}{2} - \frac{a}{2} + 1 = 3(\frac{a}{2}) + 1\frac{a+2}{2}$, we obtain $(2a+1) \in \langle \frac{a}{2}, \frac{a+2}{2}\rangle$.

Theorem 2: The Frobenius number of the telescopic numerical semigroup $S = \langle a, a+2, 2a+1\rangle$ is $g(S) = \frac{a^2}{2} + a - 1$ and $g(S)$ is odd,where $a > 2$ and a is even integer number.

Proof : Putting $a_1 = a, a_2 = a+2$ and $d = 2$ from [5, proposition 2.2],we write $g(S)) = \frac{a^2}{2} + a - 1$. On the other hand,if we get $a = 2k, k > 1$, we obtain that $g(S) = 2k^2 + 2k - 1$ and $g(S)$ is odd.

Theorem 3: Let $S = \langle a, a+2, 2a+1\rangle$. Telescopic numerical semigroup S is symmetric and the corner of $Ap(S, 2a+1)$ is $4a+2$,where $a > 2$ and a is even integer number.

Proof : It is clear that telescopic numerical semigroup S is symmetric from [5].For $S = \langle a, a+2, 2a+1\rangle$. and $u \in \langle a, a+2\rangle$, we will show that the number which satisfies the following conditions is $u = 4a+2$:

(i)$u \notin Ap(S, 2a+1)$ since $(4a+2) - (2a+1) = (2a+1) \in S$ and $(2a+1) - (2a+1) = 0 \in S$.

(ii)$(u - a) \in Ap(S, 2a+1)$ since $(4a+2) - (a) = (3a+2) \in S$ and $(3a+2) - (2a+1) = (a+1) \notin S$.

(iii) $(u-(a+2)) \in Ap(S, 2a+1)$ since $(4a+2)-(a+2)=(3a) \in S$ and $(3a)-(2a+1)=(a-1) \notin S$.

Example: Let $a=4$. In this case, we write $S = \langle 4,6,9\rangle = \{0,4,6,8,9,10,12,\rightarrow,\cdots\}$. S is telescopic numerical semigroup since $d = g.c.d\{4,6\} = 2$ and $9 \in \langle \frac{4}{2}, \frac{6}{2}\rangle = \langle 2,3\rangle$. On the other hand, we obtain $g(S)) = \frac{a^2}{2}+a-1 = 11$, and the corner of $Ap(S,9) = \{s \in S : (s-9) \notin S\} = \{0,4,6,8,10,12,16,20\}$ is 18. Finally,we find that $H(S) = \{1,2,3,5,7,11\}$ and $FH(S) = \{2,3,5,7,11\}$.

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Received: October 3, 2005