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LATTICE POINTS AND MULTIPLICATION TABLES

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It is an honor for me to dedicate this note to my dear friend Annette

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Consider the following $n \times n$ Multiplication Table, denoted by $\mathrm{MT}_{n\times n}$.

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

One of the wonderful results about $\mathrm{MT}_{n\times n}$ is Erdös Multiplication Table Theorem [1], which asserts $M(n) = o(n^2)$ when $n \to \infty$, where $M(n) = \#\{ij | 1 \leq i, j \leq n\}$. In fact $M(n)$ is the number of distinct numbers in $\mathrm{MT}_{n\times n}$. More precisely, Erdös showed that $M(n) = n^2(\log n)^{-c+o(1)}$ for $c = 1 - \frac{1+\log\log 2}{\log 2}$ [2, 3]. The following table includes some computational results about $M(n)$ by the Maple software.

n	$M(n)$	$M(n)/n^2 \approx$	n	$M(n)$	$M(n)/n^2 \approx$
10	42	0.4200000000	2000	959759	0.2399397500
50	800	0.3200000000	3000	2121063	0.2356736667
100	2906	0.2906000000	4000	3723723	0.2327326875
1000	248083	0.2480830000	5000	5770205	0.2308082000

Note that, the true order of $M(n)$ is $n^2(\log n)^{-c}(\log\log n)^{-3/2}$ [3]. Now, consider the lattice $\mathcal{L}_{2,n} = \mathbb{N}_n^2$, with $\mathbb{N}_n = \{1, 2, \cdots, n\}$. Clearly, $\mathrm{MT}_{n\times n}$ is generated by multiplying point's components in $\mathcal{L}_{2,n}$. This idea is generalizable. Consider the lattice $\mathcal{L}_{k,n} = \mathbb{N}_n^k$ in $\mathbb{R}^k$. A $k-$dimensional multiplication table, denoted by $\mathrm{MT}^k_{n\times n}$, is a $k-$dimensional array of n^k numbers in $\mathbb{R}^k$ in which every number is generated by multiplying components of corresponding lattice point in $\mathcal{L}_{k,n}$. Throughout this note, we let $\mathcal{A} = (a_1, a_2, \cdots, a_k)$.

Letting $M(k;n) = \#\big\{a_1a_2\cdots a_k|\mathcal{A} \in \mathcal{L}_{k,n}\big\}$ as a generalization of $M(n)$, we have $M(k+1;n) < nM(k;n)$, and considering this inequality with Erdös's theorem, we obtain a Generalization of Erdös Multiplication Table Theorem; $M(k;n) = o(n^k)$ when $n \to \infty$. Moreover, considering $M(n) \ll n^2(\log n)^{-c}$, we obtain $M(k;n) \ll n^k(\log n)^{-e_k}$, for some constant $e_k > 0$.
Study of the sequence $\{E_k\}_{k\geq 1}$ with $E_k = \sup\{e_k|M(k;n) \ll n^k(\log n)^{-e_k}\}$ is a nice problem; we know that $E_1 = c$ and it is increasing, because $M(k+1;n) < nM(k;n)$. Also, it seems to be diverges (we have no reason).
***Q**uestion 1.* What is the true order of $M(k;n)$?
As we saw, generalization of multiplication table based on lattice points in $\mathbb{R}^k$. But, $\mathbb{R}^k$ is a very special k-dimensional manifold. If we replace $\mathbb{R}^k$ with Υ, an $l-$dimensional manifold with $l \leq k$ and embedded in $\mathbb{R}^k$, then we can define generalized multiplication table on Υ by considering lattice points on it (this isn't easy). Let $\mathcal{L}_{\Upsilon,n} = \Upsilon \cap \mathbb{N}_n^k$, and $M(\Upsilon,n) = \#\big\{a_1a_2\cdots a_k|\mathcal{A} \in L_\Gamma(n)\big\}$. Analogue of Erdös Multiplication Table Theorem in this case is $M(\Upsilon,n) = o(\#\mathcal{L}_{\Upsilon,n})$, which is not valid for all manifolds. For example, we can make an $l-$manifold Υ_0 with $M(\Upsilon,n) = \#\mathcal{L}_{\Upsilon,n}$. To do this, consider $\mathcal{PL} = \{(p_{1+j-1}, p_{2+j-1}, \cdots, p_{k+j-1})|j = 1, 2, \cdots\}$, such that p_j denote the j^{th} prime, and take $l-$manifold Υ_0, embedded in $\mathbb{R}^k$, with $\Upsilon_0 \cap \mathbb{N}^k = \mathcal{PL}$.
***Q**uestion 2.* For which manifolds, Erdös Multiplication Table Theorem is valid? If $M(\Upsilon,n) = o(\#\mathcal{L}_{\Upsilon,n})$ holds, what is true order of $M(\Upsilon,n)$?

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