

Asymptotic methods and some difference fractional differential equations

Khairia El-Said El-Nadi

Faculty of Science, Alexandria University
Alexandria, Egypt
khairia_el_said@hotmail.com

Abstract

A probability scheme governed by a difference fractional differential equation of the form,

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \sum_{i=1}^m a_i u(x + y_i, t),$$

is studied, where $0 < \alpha \leq 1$, $x = (x_1, \dots, x_n)$, $y_i = (y_{i1}, \dots, y_{in})$ are points in the n -dimensional Euclidean space R_n and $a_i \in R_1, i = 1, \dots, m$. An asymptotic solution of the Cauchy problem for the considered equation is given.

A general grey Brownian motion and grey noises are studied by using the considered probability scheme.

Keywords: Difference fractional differential equations, Grey Brownian motion.

Mathematics Subject Classification: 35 J 05, 35 J 10, 35 K 05.

1. Introduction

Let X_1, \dots, X_n be independent identically distributed nonnegative random variables. Suppose that G_α is the distribution of $Y_n = (\frac{1}{n})^\frac{1}{\alpha} \sum_{r=1}^n X_r$ and also every

random variable $X_r, r = 1, \dots, n$. In this case G_α is called a strictly stable distribution function, (see Schneider (1986) [10], Zolotarev 1986 [16] and Uchaikin 2000 [14]).

$$\text{If } \lim_{x \rightarrow \infty} x^\alpha [1 - G_\alpha(x)] = \frac{1}{\Gamma(1 - \alpha)}, \quad \lim_{x \rightarrow 0} e^{-x^\alpha} G_\alpha(x) = 0,$$

where $0 < \alpha < 1$, then the Laplace transform of G_α is given by

$$\int_0^\infty e^{-\lambda x} G_\alpha(x) dx = e^{-\lambda^\alpha},$$

and the Laplace transform of

$$\xi_\alpha(x) = 1 - G_\alpha\left(\left(\frac{1}{x}\right)^\frac{1}{\alpha}\right),$$

is given by

$$\int_0^\infty e^{-\lambda x} \xi_\alpha(x) dx = F_\alpha(\lambda),$$

where F_α is the Mittag-Leffler function;

$$F_\alpha(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(1 + k\alpha)}$$

and $\Gamma(\alpha)$ is the gamma function. Using the properties of the probability density function ξ_α , we define a generalized Poisson stochastic process, whose density function is given by

$$p_K(t) = \int_0^\infty \xi_\alpha(\theta) \frac{(At^\alpha\theta)^K}{K!} \exp(-t^\alpha\theta S) d\theta, \quad (1.1)$$

where $K = (k_1, \dots, k_m)$, k_1, \dots, k_m are nonnegative integers, $K! = k_1! \dots k_m!$, $A = (a_1, \dots, a_m)$, a_1, \dots, a_m are positive members, $A^K = a_1^{k_1} \dots a_m^{k_m}$ and $S = a_1 + \dots + a_m$.

In section 2, we shall consider a probability scheme governed by the following system of equations;

$$\frac{d^\alpha P_K(t)}{dt^\alpha} = -SP_K(t) + \sum_{i=1}^n a_i P_{K-e_i}(t) + \sum_{i=1}^r a_{n+i} P_{K+g_i}(t), \quad (1.2)$$

where e_1, \dots, e_n are the unit vectors; $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1) \in R_n$, R_n is the n-dimensional Euclidean space, $g_i = (b_{i1}, \dots, b_{in})$, and b_{ij} are

integers, $i = 1, \dots, r, j = 1, \dots, n$. We suppose that $P_K(t)$ satisfies the initial conditions

$$P_K(0) = \begin{cases} 1 & \text{at } K = (0, \dots, 0) \\ 0 & \text{at } K \neq (0, \dots, 0) \end{cases} \quad (1.3)$$

The initial value problem (1.2), (1.3) is solved in terms of the probability density functions $p_K(t)$.

Let $B_\alpha(t), t > 0, 0 < \alpha \leq 1$ be the grey Brownian motion. In other words $B_\alpha(t)$ is a stochastic process whose characteristic function is given by

$$E[\exp\{i\lambda(B_\alpha(t) - B_\alpha(s))\}] = F_\alpha(\lambda^2(t - s)^\alpha), \quad 0 \leq s \leq t.$$

For $\alpha = 1$ the Brownian motion is discovered.

It is known that the solution of the initial value problem;

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u(x, t)}{\partial x^2},$$

$-\infty < x < \infty, t > 0, u(x, 0) = f(x)$, is given by

$$u(x, t) = E(f(x + B_\alpha(t))),$$

for sufficiently well behaved function f , (Schneider 1990 [12]).

The stochastic process $B_\alpha(t)$ is constructed on a probability measure called grey noise.

In section 3, we shall generalize the previous results, (see El-Nadi 1969, 1970 [1], [2], Mainardi 2001 [8] and Uchaikin 2000 [14]).

Following Caputo, the fractional derivative of order α is defined by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \theta)^{-\alpha} \frac{df(\theta)}{d\theta} d\theta,$$

$0 \leq \alpha < 1$ and the fractional integral of order α is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha-1} f(\theta) d\theta,$$

$0 < \alpha \leq 1$, (see Schneider 1989 [11], Wyss 1986 [15], and Nigmatullin 1992 [9]).

2. A grey Brownian motion

Let e_{n+1}, \dots, e_{n+r} be vectors in R_n such that

$$BE = 0, \quad (2.1)$$

where B and E are the matrices $B = [(b_{ij})|I], i = 1, \dots, r, j = 1, \dots, n, I$ is the unit n by n matrix and $E = [e_1 \dots e_n \dots e_{n+r}]^T$, (A^T is the transpose of the matrix A).

We shall use the generalized barycentric coordinates, (see El-Nadi 1983 [3]), Ljjesternik 1967 [6] and Samorodnitsky 1994 [13]). We resolve the vector K into the generalized barycentric coordinates with respect to the vectors e_1, \dots, e_{n+r} ;

$$K = \sum_{i=1}^m \mu_i e_i, m = n + r \quad (2.2)$$

It is clear that the vector K has infinitely many representations by the generalized barycentric coordinates (μ_1, \dots, μ_m) . Using (2.1), we get

$$e_{n+i} = -(b_{i1}e_1 + b_{i2}e_2 + \dots + b_{in}e_n), \quad i = 1, \dots, r \quad (2.3)$$

From (2.3) it is easy to get

$$K + g_i = K - e_{n+i} \quad (2.4)$$

Now equation (1.2) can be written in the form

$$\frac{d^\alpha P_K(t)}{dt^\alpha} = -SP_K(t) + \sum_{i=1}^m a_i P_{K-e_i}(t). \quad (2.5)$$

To take the initial condition $P_K(0)$ automatically into account, it is convenient to consider the definition of fractional integral. In this case the initial value problem (1.2), (1.3) can be written in the form

$$P_K(t) = P_K(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} Q_K(\theta) d\theta, \quad (2.6)$$

where

$$Q_K(t) = -SP_K(t) + \sum_{i=1}^m a_i P_{K-e_i}(t).$$

Using a previous result of the author (El-Nadi 1983 [3]) we find that the unique solution of the fractional difference equation (2.6) is given by;

$$P_K(t) = \int_0^\infty \xi_\alpha(\theta) \sum_q \frac{(t^\alpha A\theta)^{\mu+(\gamma,q)}}{(\mu + (\gamma, q))!} \exp[-t^\alpha \theta S] d\theta, \quad (2.7)$$

where $\mu = (\mu_1, \dots, \mu_m)$, $\gamma = (\gamma_1, \dots, \gamma_n)$, $q = (q_1, \dots, q_r)$, $(\gamma, q) = (\gamma_1, \dots, \gamma_n, q_1, \dots, q_r)$, $\gamma_j = \sum_{i=1}^r b_{ij} q_i$, $i = 1, \dots, r$, $\sum_q = \sum_{q_1} \dots \sum_{q_r}$, $\sum_{q_j} = \sum_{q_j=-\infty}^\infty$, $\frac{1}{\gamma_j!} = 0$ if $\gamma_j < 0$, (see El-Borai 2002 [4]).

Using (2.3) and (2.4), we can write (2.7) in the form

$$P_K(t) = \int_0^\infty \xi_\alpha(\theta) \sum_q \frac{(t^\alpha \theta A_1)^{K+\gamma}}{(K + \gamma)!} \frac{(t^\alpha \theta A_2)^q}{q!} \exp[-t^\alpha \theta S] d\theta,$$

where $A_1 = (a_1, \dots, a_n)$, $A_2 = (a_{n+1}, \dots, a_{n+r})$.

If $V_K(t)$ satisfies equation (1.2) and the initial condition

$$V_K(0) = b_K, \quad (2.8)$$

then the unique solution of the problem (1.2), (2.8) is given by

$$V_K(t) = \sum_M P_{K-M}(t) b_M, \quad (2.9)$$

where $M = (m_1, \dots, m_n)$, m_1, \dots, m_n are integers.

It is easy now to deduce that

$$\sum_K P_K(t) = 1$$

The problem (1.2), (1.3) and formula (2.7) describe a grey Brownian motion in R_n beginning at the origin of coordinates and have jumps, which form a generalized Poisson process defined by (1.1), (see Montroll 1984 [7]).

3. Asymptotic expansions

We shall find an asymptotic expansion for $V_K(t)$ and then for the probability density function $P_K(t)$.

Let us consider the following problem

$$h^2 \frac{\partial^\alpha u(x, t, h)}{\partial t^\alpha} = \sum_{j=0}^n a_j^2 [u(x - h e_j, t, h) - u(x, t, h)], \quad (3.1)$$

$$u(x, 0, h) = \phi(x), \quad (3.2)$$

where $x \in R_n, t > 0, a_j > 0, j = 0, 1, \dots, n, \phi(x)$ is a given infinitely differentiable function with bounded support, e_1, \dots, e_n constitute a basis in R_n and e_0 is defined by

$$e_0 + e_1 + \dots + e_n = 0. \quad (3.3)$$

We resolve the vector x into the barycentric coordinates with respect to the vectors e_0, e_1, \dots, e_n ;

$$x = \sum_{j=0}^n \frac{y_j}{a_j} e_j, \quad (3.4)$$

Using (3.3) and (3.4), we get

$$\sum_{j=0}^n a_j \frac{\partial u}{\partial y_j} = 0. \quad (3.5)$$

With the help of (3.5), equation (3.1) can be written in the form

$$\begin{aligned} \frac{\partial^\alpha u(x, t, h)}{\partial t^\alpha} &= \frac{1}{2} \sum_{j=0}^n \frac{\partial^2 u(x, t, h)}{\partial y_j^2} - \frac{h}{3!} \sum_{j=0}^n a_j^{-1} \frac{\partial^3 u(x, t, h)}{\partial y_j^3} + \dots + \\ &+ \frac{(-1)^{k+1} h^{k-1}}{(k+1)!} \sum_{j=0}^n a_j^{-k+1} \frac{\partial^{k+1} u(x, t, h)}{\partial y_j^{k+1}} + R(x, t, h). \end{aligned} \quad (3.6)$$

We shall write $u \in C_y^\infty$ if u has bounded derivatives of all orders with respect to y on R_{n+1} . If $u \in C_y^\infty$, then $R(x, t, h) = O(h^k)$.

We shall use the following notations;

$$\gamma_1 = \sum_{j=0}^n a_j, \quad \gamma_2 = \sum_{j=0}^n a_j^2, \quad \gamma = \prod_{j=0}^n a_j, \quad \psi(t) = \frac{\gamma}{\sqrt{(2\pi t)^n \gamma_2}},$$

$$\delta_j(t) = \left[\frac{\gamma_2}{2t(\gamma_2 - a_j^2)} \right]^{\frac{1}{2}},$$

$$\eta'_j = a_j [\alpha_j \eta_j - \frac{1}{\gamma_2} (a_1^2 \eta_1 + \dots + a_n^2 \eta_n)] \quad , \quad q_j = \left(\frac{\gamma_2 - a_j^2}{\gamma_2} \right) y_j - \frac{a_j}{\gamma_2} \sum_{r \neq j} a_r y_r,$$

$$\eta_j^* = q_j - a_j \eta'_j \quad , \quad \alpha_0 = 0 \quad , \quad \alpha_1 = \alpha_2 = \dots = \alpha_n = 1.$$

Theorem 3.1. For sufficiently small $h > 0$, the solution of the initial value problem (3.1), (3.2) has the following asymptotic representation;

$$u(x, t, h) = u_0(x, t) + \sum_{j=1}^{k-1} h^j u_j(x, t) + Z(x, t, h),$$

where $u \in C_y^\infty$, $\frac{\partial^\alpha u}{\partial t^\alpha}$ is continuous on $R_{n+1} \cup [0, \infty)$, $Z \in C_y^\infty$, $Z(x, t, h) = O(h^k)$, $u_0(x, 0) = \phi(x)$, $u_j(x, 0) = Z(x, 0, h) = 0$, $j = 1, \dots, k-1$,

$$u_0(x, t) = \int_0^\infty \int_{R_n} \psi(t^\alpha \theta) \xi_\alpha(\theta) \phi(\eta) \exp \left[\frac{-1}{2t^\alpha \theta} \sum_{j=0}^n \eta_j^{*2} \right] d\eta d\theta, \quad (3.7)$$

$$u_1(x, t) =$$

$$\frac{-t^\alpha}{3!} \int_0^\infty \int_{R_n} \theta \psi(t^\alpha \theta) \xi_\alpha(\theta) \phi(\eta) \sum_{j=0}^n a_j^{-1} H_3^*(\delta_j(t^\alpha \theta) \eta_j^*) \exp \left[\frac{-1}{2t^\alpha \theta} \sum_{j=0}^n \eta_j^{*2} \right] d\eta d\theta,$$

$$H_r^*(\delta_j(t) q_j) = (-1)^r \left[\frac{\gamma_2 - a_j^2}{2t\gamma_2} \right]^{r/2} H_r(\delta_j(t) q_j),$$

$H_r(t) = (-1)^r \exp(t^2) \frac{d^r}{dt^r} \exp(-t^2)$ are the Hermite polynomials.

All the functions u_2, \dots, u_{k-1} can be determined as a linear combinations of

$$t^{\alpha p} \frac{\partial^\beta u_0}{\partial y_{i_1}^{\beta_1} \dots \partial y_{i_r}^{\beta_r}} \quad \text{where } \beta = \beta_1 + \dots + \beta_r$$

$0 < p < \beta/2$, $(\beta_1, \dots, \beta_r$ are nonnegative integers).

Proof. The proof of the required asymptotic formula can be carried out by using the method of small parameters. Since some parts of this proof is similar to that made by the author (El-Nadi 1969, 1983, [1], [3]), we shall avoid some details.

Problem (3.1), (3.2) can be written in the form

$$u(x, t, h) = \phi(x) + \frac{1}{h^2 \Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha-1} \nu(x, \theta, h) d\theta, \quad (3.8)$$

where

$$\nu(x, t, h) = \sum_{j=0}^n a_j^2 [u(x - h e_j, t, h) - u(x, t, h)].$$

Let $\omega(x, t, h)$ be the solution of (3.8) at $\alpha = 1$. We try to find ω in the form

$$\omega(x, t, h) = \omega_0(x, t) + \sum_{j=1}^{k-1} h^j \omega_j(x, t) + Z_k^*(x, t, h),$$

where

$$L_0 \omega_0(x, t) = 0,$$

$$L_0 \omega_s = -\frac{1}{3!} \sum_{j=0}^n a_j^{-1} \frac{\partial^3 \omega_{s-1}(x, t)}{\partial y_j^3} + \dots + \frac{(-1)^s}{(s+2)!} \sum_{j=0}^n a_j^{-s} \frac{\partial^{s+2}}{\partial y_j^{s+2}} \omega_0(x, t), \quad s = 1, \dots, k-1,$$

$$L_0 = \frac{\partial}{\partial t} - \frac{1}{2} \sum_{j=0}^n \frac{\partial^2}{\partial y_j^2},$$

$$\omega_0(x, 0) = \phi(x) \quad , \quad \omega_1(x, 0) = \dots = \omega_{k-1}(x, 0) = 0 \quad , \quad Z^*(x, 0, h) = 0.$$

It is clear that

$$\omega_0(x, t) = \frac{1}{\sqrt{(2\pi t)^{n+1}}} \int_{R_{n+1}} \phi \left(\sum_{j=0}^n \frac{\xi_j}{a_j} e_j \right) \exp \left[\frac{-1}{2t} \sum_{j=0}^n (y_j - \xi_j)^2 \right] d\xi,$$

where $d\xi = d\xi_0 d\xi_1 \dots d\xi_n$

Set $\eta_0 = \frac{\xi_0}{a_0}$, $\eta_j = \frac{\xi_j}{a_j} - \eta_0$, $g_0 = y_0$, $g_j = y_j - a_j \eta_j$, $j = 1, \dots, n$, $\bar{g} = \frac{1}{\gamma_1} \sum_{j=0}^n a_j g_j$.

Since $\sum_{j=0}^n (y_j - \xi_j)^2 = \gamma_2 (\eta_0 - \frac{\gamma_1}{\gamma_2} \bar{g})^2 + \sum_{j=0}^n g_j^2 - \frac{\gamma_1^2}{\gamma_2} \bar{g}^2$, ω_0 can be written in the form;

$$\omega_0(x, t) = \psi(t) \int_{R_n} \phi(\eta) \exp \left[\frac{-1}{2t} \sum_{j=0}^n (g_j - \frac{\gamma_1}{\gamma_2} a_j \bar{g})^2 \right] d\eta,$$

where $\phi(\eta) = \phi(\sum_{j=1}^n \eta_j e_j)$.

Noticing that $g_j - \frac{\gamma_1}{\gamma_2} a_j \bar{g} = q_j - a_j \eta'_j$, $j = 0, 1, \dots, n$, we get

$$\omega_0(x, t) = \psi(t) \int_{R_n} \phi(\eta) \exp \left[\frac{-1}{2t} \sum_{j=0}^n (\eta_j^*)^2 \right] d\eta$$

It is easy to see that ω_1 can be given in the form

$$\begin{aligned} \omega_1(x, t) &= \frac{-t}{3!} \sum_{j=0}^n a_j^{-1} \frac{\partial^3 \omega_0(x, t)}{\partial y_j^3} \\ &= \frac{-t}{3!} \psi(t) \int_{R_n} \phi(\eta) \sum_j a_j^{-1} H_3^*[\delta_j(t) \eta_j^*] \exp \left[\frac{-1}{2t} \sum_{j=0}^n (\eta_j^*)^2 \right] d\eta, \end{aligned}$$

In the same way, we find

$$\begin{aligned} \omega_2(x, t) &= \frac{t^2}{2!(3!)^2} \psi(t) \int_{R_n} \phi(\eta) \sum_{i,j} \bar{H}_{3i,3j} \exp \left[\frac{-1}{2t} \sum_{j=0}^n (\eta_j^*)^2 \right] d\eta \\ &+ \frac{t}{4!} \psi(t) \int_{R_n} \phi(\eta) \sum_j a_j^{-2} H_4^*(\delta_j(t) \eta_j^*) \exp \left[\frac{-1}{2t} \sum_{j=0}^n (\eta_j^*)^2 \right] d\eta, \end{aligned}$$

where

$$\bar{H}_{3i,3j} = \begin{cases} H_3^*(\delta_i(t) \eta_i^*) H_3^*(\delta_j(t) \eta_j^*) & , i \neq j \\ H_6^*(\delta_j(t) \eta_j^*) & , i = j \end{cases}$$

Since ϕ is a test function, it follows that all the functions $\omega_0, \omega_1, \dots, \omega_{k-1}$ belong to C_y^∞ , (Gelfand 1968 [5]).

Using the Fourier transform, we can show that $\omega \in C_y^\infty$. Thus $Z^* \in C_y^\infty$. We notice that the remainder Z^* satisfies the equation

$$\begin{aligned} \frac{\partial Z^*(x, t, h)}{\partial t} &= \sum_{j=0}^n a_j^2 [Z^*(x - he_j, t, h) - Z^*(x, t, h)] \\ &= O(h^k), Z^*(x, 0, h) = 0. \end{aligned}$$

Thus using Duhamel's principle and the Fourier transform, we can prove that $Z^*(x, t, h) = O(h^k)$ (see El-Nadi 1969, 1983 [1], [3]).

Using the fact that

$$u(x, t, h) = \int_0^\infty \xi_\alpha(\theta) \omega(x, t^\alpha \theta, h) d\theta,$$

(see El-Borai 2002 [4]), we get

$$\begin{aligned} u(x, t, h) &= \int_0^\infty \xi_\alpha(\theta) \omega_0(x, t^\alpha \theta) d\theta + \\ &+ \sum_{j=1}^{k-1} \int_0^\infty \xi_\alpha(\theta) h^j \omega_j(x, t^\alpha \theta) d\theta + O(h^k). \end{aligned} \quad (3.9)$$

This completes the proof.

Consider the function $P_\mu(t)$ defined by

$$P_\mu(t) = \int_0^\infty \xi_\alpha(\theta) \sum_{m=-\infty}^\infty \frac{(t^\alpha \theta A)^{\mu+M}}{(\mu+M)!} \exp[-t^\alpha \theta S] d\theta,$$

where $\mu = (\mu_0, \mu_1, \dots, \mu_n)$, $M = (m, m, \dots, m)$, $A = (a_0^2, a_1^2, \dots, a_n^2)$ and $S = a_0^2 + a_1^2 + \dots + a_n^2$

If $K = e_0 \mu_0 + e_1 \mu_1 + \dots + e_n \mu_n$, then

$P_K = P_\mu$, where $K = (k_1, \dots, k_n)$, $k_j = \mu_j - \mu_0$, $j = 1, \dots, n$

It is clear that $P_\mu(t)$ satisfies the equation;

$$P_\mu(t) = P_\mu(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} Q_\mu(\theta) d\theta,$$

where

$$Q_\mu(t) = \sum_{j=0}^n a_j^2 P_{\mu-e_j}(t) - S P_\mu(t),$$

Setting $x = \sum_{j=0}^n \frac{y_j}{a_j} e_j = \sum_{j=0}^n h \mu_j e_j$, $b_\mu = \phi(\sum_{j=0}^n h \mu_j e_j)$, we deduce that $V_\mu(t) = \sum_\gamma P_{\mu-\gamma}(\frac{t}{h^{2/\alpha}}) b_\gamma$ represents a solution of the problem (3.1), (3.2). Setting

$\tau = \frac{1}{h^2}$, $t = 1$, we obtain the following asymptotic formula, (with the help of Lagrange's lemma and Euler's formula and (3.9));

$$\begin{aligned} P(\tau^\alpha) &= \gamma \psi(\tau) \int_0^\infty s^{n/2} \xi_\alpha^*(s) \exp \left[\frac{-s}{2\tau} \sum_{j=1}^n k_j^{*2} \right] ds \\ &- \frac{\gamma \psi(\tau)}{3!} \left(\frac{1}{\tau}\right)^{1/2} \int_0^\infty s^{\frac{n}{2}-1} \xi_\alpha^*(s) \sum_{j=0}^n a_j^{-1} H_3^* \left(\frac{\gamma_2 s k_j^*}{\gamma_2 - a_j^2} \right) \exp \left[\frac{-s}{2\tau} \sum_{j=1}^n k_j^{*2} \right] ds \\ &+ O\left(\frac{1}{\tau}\right), \end{aligned}$$

for sufficiently large τ , where

$$\begin{aligned} \xi^*(s) &= \frac{1}{s^2} \xi\left(\frac{1}{s}\right) \quad , \quad \left(\int_0^\infty \xi^*(s) ds = 1 \right), \\ k_j^* &= a_j [a_j \alpha_j k_j - \frac{1}{\gamma_2} (a_1^3 k_1 + \dots + a_n^3 k_n)]. \end{aligned}$$

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Received: September 23, 2005