FEKETE-SZEGÖ PROBLEM
FOR CERTAIN SUBCLASS
OF QUASI-CONVEX FUNCTIONS

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Abstract

For $0 \leq \alpha < 1$, let $Q_\alpha$ be the class of functions $f$ which are normalised analytic and univalent in $D = \{ z : |z| < 1 \}$ satisfying the condition

$$
\text{Re} \left\{ \frac{\alpha(z^2f''(z))'}{g'(z)} + \frac{(zf'(z))'}{g'(z)} \right\} > 0,
$$

where $g$ is a normalised convex function. For $f \in Q_\alpha$, sharp bounds are obtained for the Feketo-Szegö functional $|a_3 - \mu a_2^2|$ when $\mu$ is real.

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1. Introduction

Let $S$ denote the class of normalised analytic univalent functions $f$ of the form

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
$$

(1)
where \( z \in D = \{ z : |z| < 1 \} \). We also denote by \( S^*, C \) and \( K \) the subclasses of \( S \) consisting of functions which are, respectively, starlike, convex and close-to-convex in \( D \).

A classical result of Feketo and Szegö [2] determines the maximum value of \( |a_3 - \mu a_2^2| \), as a function of the real parameter \( \mu \), for functions belonging to \( S \). There are now several results of this type in the literature, each of them dealing with \( |a_3 - \mu a_2^2| \) for various classes of functions (see, e.g., [1,4]).

Denote by \( Q(\beta) \) the class of strongly quasi-convex functions of order \( \beta (\beta \geq 0) \). Thus \( f \in Q(\beta) \) if and only if there exists \( g \in C \) such that for \( z \in D \),

\[
\left| \arg \left\{ \frac{(zf'(z))^'}{g'(z)} \right\} \right| \leq \frac{\pi \beta}{2}.
\]

In particular, \( Q = Q(1) \) is the class of quasi-convex functions introduced by Noor [7]. We also note that every quasi-convex function is close-to-convex and hence univalent in \( D \). For functions belonging to the class \( Q(\beta) \), sharp upper bounds for the functional \( |a_3 - \mu a_2^2| \) have been obtained by Nak Eun Cho [6].

In this paper, we give an estimate for the same functional for the class \( Q_\alpha \) defined as follows:

**Definition 1** Let \( f \) be given by (1) and \( 0 \leq \alpha < 1 \). Then \( f \in Q_\alpha \) if and only if there exist \( g \in C \) such that for \( z \in D \),

\[
Re \left\{ \frac{\alpha(z^2 f''(z))'}{g'(z)} + \frac{(zf'(z))^'}{g'(z)} \right\} > 0. \tag{2}
\]

Here, \( C \) denotes the class of convex functions; that is \( g \in C \) if and only if \( g \) is analytic in \( D \) and

\[
Re \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > 0 \tag{3}
\]

for \( z \in D \).

We note that by using a lemma due to Miller and Mocanu [5], it can easily be shown that \( Q_\alpha \subset Q \) for \( 0 \leq \alpha < 1 \) and hence \( f \in Q_\alpha \) means \( f \) is univalent.

We first state some preliminary lemmas, required for proving our result.

### 2. Preliminary Results

**Lemma 1** ([8]) Let \( h \) be analytic in \( D \) with \( Re \, h(z) > 0 \) and be given by \( h(z) = 1 + c_1 z + c_2 z^2 + ... \) for \( z \in D \), then
\[ \left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}. \]

**Lemma 2** ([3]) Let \( g \in \mathbb{C} \) with \( g(z) = z + b_2 z^2 + b_3 z^3 + ... \) Then, for \( \mu \) real

\[ |b_3 - \mu b_2^2| \leq \max \left\{ \frac{1}{3}, |\mu - 1| \right\}. \]

**Lemma 3** Let \( f \in \mathcal{Q}_\alpha \) and be given by (1) then

\[ (\alpha + 1)|a_2| \leq 1 \]

and

\[ (2\alpha + 1)|a_3| \leq 1. \]

**Proof.**

Since \( g \in \mathbb{C} \), it follows from (3) that

\[ g'(z) + zg''(z) = g'(z)p(z) \quad (4) \]

for \( z \in \mathcal{D} \), with \( Re \ p(z) > 0 \) given by \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \) Equating coefficients, we obtain

\[ 2b_2 = p_1 \quad (5) \]

and

\[ 6b_3 = p_2 + 2b_2 p_1. \quad (6) \]

It also follows from (2) that

\[ \alpha (z^2 f''(z))' + (z f'(z))' = g'(z)h(z) \quad (7) \]

where \( Re \ h(z) > 0 \). Writing \( h(z) = 1 + c_1 z + c_2 z^2 + ... \) and equating coefficients in (7) gives

\[ 4(\alpha + 1)a_2 = c_1 + 2b_2 \quad (8) \]

and

\[ 9(2\alpha + 1)a_3 = c_2 + 2b_2 c_1 + 3b_3. \quad (9) \]

The result now follows on using classical inequalities \( |p_1| \leq 2, |p_2| \leq 2, |c_1| \leq 2, |c_2| \leq 2 \) and the inequalities \( |b_2| \leq 1 \) and \( |b_3| \leq 1 \) which follow from (5) and (6).
3. Main Result

**Theorem.** Let \( f \) be given by (1) and belongs to the class \( Q_\alpha \). Then, for \( 0 \leq \alpha < 1 \),

\[
9(2\alpha + 1)(\alpha + 1)^2|a_3 - \mu a_2^2|
\]

\[
\leq \begin{cases} 
9(\alpha + 1)^2 - 9(2\alpha + 1)\mu , & \text{if } \mu \leq \frac{4(\alpha+1)^2}{9(2\alpha+1)}, \\
5(\alpha + 1)^2 - \frac{9(2\alpha+1)\mu}{(8(\alpha+1)^2-9(2\alpha+1)\mu)^2} + \frac{4}{36(2\alpha+1)\mu} , & \text{if } \frac{4(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{8(\alpha+1)^2}{9(2\alpha+1)}, \\
3(\alpha + 1)^2 , & \text{if } \frac{8(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{4(\alpha+1)^2}{3(2\alpha+1)}, \\
-9(\alpha + 1)^2 + 9(2\alpha + 1)\mu , & \text{if } \mu \geq \frac{4(\alpha+1)^2}{3(2\alpha+1)}. 
\end{cases}
\]

Inequalities are sharp for all cases.

**Proof.**

From (5),(7),(8) and (9), it is easily established that

\[
9(2\alpha + 1)(a_3 - \mu a_2^2)
\]

\[
= 3 \left\{ b_3 - \frac{3(2\alpha + 1)\mu b_2^2}{4(\alpha + 1)^2} \right\} + \left\{ c_2 + \left( \frac{8(\alpha + 1)^2 - 9(2\alpha + 1)\mu}{16(\alpha + 1)^2} - \frac{1}{2} \right) c_1^2 \right\} + \left\{ 1 - \frac{9(2\alpha + 1)\mu}{8(\alpha + 1)^2} \right\} p_1 c_1. 
\]

(10)

First, consider the case \( \frac{4(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{8(\alpha+1)^2}{9(2\alpha+1)} \).

Equation (10) gives

\[
9(2\alpha + 1)|a_3 - \mu a_2^2| \leq 3 \left| b_3 - \frac{3(2\alpha + 1)\mu b_2^2}{4(\alpha + 1)^2} \right| + \left| c_2 - \frac{1}{2} c_1^2 \right| + \frac{1}{16(\alpha+1)^2}|8(\alpha + 1)^2 - 9(2\alpha + 1)\mu||c_1||
\]

\[
+ \frac{1}{8(\alpha+1)^2}|8(\alpha + 1)^2 - 9(2\alpha + 1)\mu||c_1||p_1|
\]

\[
\leq \left( 3 - \frac{9(2\alpha + 1)\mu}{4(\alpha + 1)^2} \right) \left( 2 - \frac{1}{2}|c_1|^2 \right) + \frac{1}{16(\alpha+1)^2}|8(\alpha + 1)^2 - 9(2\alpha + 1)\mu||c_1|
\]

\[
+ \frac{1}{4(\alpha+1)^2}|8(\alpha + 1)^2 - 9(2\alpha + 1)\mu||c_1|
\]
where we have used Lemma 1 and Lemma 2 and the inequality $|p_1| \leq 2$. Elementary calculation indicates that the function $\varphi$ attains its maximum value at $x_o = \frac{2(8(\alpha+1)^2 - 9(2\alpha+1)\mu)}{9(2\alpha+1)\mu}$ and thus establishing

$$9(2\alpha + 1)(\alpha + 1)^2|a_3 - \mu a_2^2| \leq \varphi(x_o)$$

Next, since $|x_o| \leq 2$, thus we have $\mu \geq \frac{4(\alpha+1)^2}{9(2\alpha+1)}$ and hence completing the proof for the case $\frac{4(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{8(\alpha+1)^2}{9(2\alpha+1)}$.

Letting $c_1 = \frac{2(8(\alpha+1)^2 - 9(2\alpha+1)\mu)}{9(2\alpha+1)\mu}$, $c_2 = p_1 = p_2 = 2$ and $b_2 = b_3 = 1$ in (10) shows that the result is sharp.

Secondly, we consider the case $\mu \leq \frac{4(\alpha+1)^2}{9(2\alpha+1)}$.

Write

$$a_3 - \mu a_2^2 = a_3 - \frac{4(\alpha+1)^2}{9(2\alpha+1)}a_2^2 + \left(\frac{4(\alpha+1)^2}{9(2\alpha+1)} - \mu\right)a_2^2.$$  

Since $|a_2| \leq \frac{1}{\alpha+1}$, it follows that

$$9(2\alpha + 1)(\alpha + 1)^2|a_3 - \mu a_2^2| \leq 9(2\alpha + 1)(\alpha + 1)^2\left|a_3 - \frac{4(\alpha+1)^2}{9(2\alpha+1)}a_2^2\right|$$

$$+ 9(2\alpha + 1)(\alpha + 1)^2\left(\frac{4(\alpha+1)^2}{9(2\alpha+1)} - \mu\right)\left(\frac{1}{\alpha+1}\right)^2 \leq 9(\alpha + 1)^2 - 9(2\alpha + 1)\mu.$$  

Here, we use the result already proven for $\mu = \frac{4(\alpha+1)^2}{9(2\alpha+1)}$. Equality is attained on choosing $c_1 = c_2 = p_1 = p_2 = 2$ and $b_2 = b_3 = 1$ in (10).

Next, assume that $\frac{8(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{4(\alpha+1)^2}{3(2\alpha+1)}$.

First, we deal with the case $\mu = \frac{4(\alpha+1)^2}{3(2\alpha+1)}$. It follows from (4),(5),(6) and (10) that

$$9(2\alpha + 1)(\alpha + 1)^2\left|a_3 - \frac{4(\alpha+1)^2}{3(2\alpha+1)}a_2^2\right| \leq 3(\alpha + 1)^2 - \frac{(\alpha+1)^2}{4}(|c_1| - |p_1|)^2,$$

$$= \psi(|c_1|, |p_1|), \text{ say.}$$
A straightforward calculation shows that the $\psi$ attains maximum value when $|c_1| = |p_1|$ and so

\[ 9(2\alpha + 1)(\alpha + 1)^2|a_3 - \mu a_2^2| \leq 3(\alpha + 1)^2. \]

Next, write

\[
a_3 - \mu a_2^2 = \frac{9(2\alpha + 1)\mu - 8(\alpha + 1)^2}{4(\alpha + 1)^2} \left( a_3 - \frac{4(\alpha + 1)^2}{3(2\alpha + 1)} a_2^2 \right) + \frac{3(4(\alpha + 1)^2 - 3(2\alpha + 1)\mu)}{4(\alpha + 1)^2} \left( a_3 - \frac{8(\alpha + 1)^2}{9(2\alpha + 1)} a_2^2 \right),
\]

and the result follows at once by using results already established for $\mu = \frac{8(\alpha + 1)^2}{9(2\alpha + 1)}$ and $\mu = \frac{4(\alpha + 1)^2}{3(2\alpha + 1)}$ above. The result is sharp for $p_2 = c_2 = 2, p_1 = c_1 = 0, b_2 = 0$ and $b_3 = \frac{1}{3}$ in (10).

Finally, consider $\mu \geq \frac{4(\alpha + 1)^2}{3(2\alpha + 1)}$.

Write

\[
a_3 - \mu a_2^2 = a_3 - \frac{4(\alpha + 1)^2}{3(2\alpha + 1)}a_2^2 + \left( \frac{4(\alpha + 1)^2}{3(2\alpha + 1)} - \mu \right) a_2^2
\]

and thus

\[
9(2\alpha + 1)(\alpha + 1)^2|a_3 - \mu a_2^2| \leq 9(2\alpha + 1)(\alpha + 1)^2 \left| a_3 - \frac{4(\alpha + 1)^2}{3(2\alpha + 1)} a_2^2 \right| + 9(2\alpha + 1)(\alpha + 1)^2 \left( \mu - \frac{4(\alpha + 1)^2}{3(2\alpha + 1)} \right) |a_2|^2,
\]

\[
\leq -9(\alpha + 1)^2 + 9(2\alpha + 1)\mu,
\]

where results for $\mu = \frac{4(\alpha + 1)^2}{3(2\alpha + 1)}$ and the inequality $|a_2| \leq \frac{1}{\alpha + 1}$ have been used.

By choosing $c_1 = p_1 = 2i, c_2 = p_2 = -2, b_2 = i$ and $b_3 = -1$ in (10), equality is obtained.

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**References**


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