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Determining the Signature of Normal Subgroups of Discrete Groups[1]

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Abstract

In this work, subgroups of a special class of discrete subgroups of $PSL(2,\mathbb{R})$, namely the ones with genus 0, have been studied. We establish a technique to compute the signature of these subgroups in terms of the signatures of easier groups. The method established here can be used for triangle groups, surface groups and Hecke groups (including the well-known modular group).

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1 Introduction

In this work, we consider an important class of discrete groups, namely those of the first kind with genus 0. A discrete group Γ is of the first kind if and only if its limit set is $\mathbb{R}$. These might be classified into three classes:

Those with elliptic elements but parabolics,

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Those with parabolic elements but elliptics,

Those with both elliptics and parabolics.

Some examples of those are the triangle groups, surface groups and Hecke groups (Including the well-known modular group) respectively.

If one defines

$$\mu(\Gamma) = 2g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + t,$$

where g is the genus of the underlying Riemann surface, t is the parabolic class number and m_i are the periods of Γ, then $2\pi\mu(\Gamma)$ is the hyperbolic area of a fundamental region of the group. Let Γ_1 be a subgroup of Γ of finite index. Then

$$[\Gamma : \Gamma_1] = \frac{\mu(\Gamma_1)}{\mu(\Gamma)}$$

is known as the Riemann-Hurwitz formula (RHF).

Now let Θ be an epimorphism between two such groups:

$$\Theta : \Gamma \longrightarrow \Delta.$$

Let Λ be a subgroup of Δ having genus g. We are interested in finding the genus of the inverse image group $\Theta^{-1}(\Lambda)$ of Λ in terms of g. The group Δ is usually "simpler" than Γ. Therefore by means of the RHF, it is easier to find the genus g of Λ rather than the genus, say g', of any subgroup of Γ. For this reason, we shall use the inverse image of Λ and hence g.

To use the RHF, one needs to know the periods of Λ and $\Theta^{-1}(\Lambda)$. One way of doing this is to make use of a result of D. Singerman, [4]. The original form of this theorem applies to all Fuchsian groups, but here, as we noted earlier, we restrict ourselves to the ones of the first kind. It is sometimes convenient to consider the parabolic elements as elliptic elements of infinite order. So we can assume that a group Γ_2 has signature $(g; m_1, ..., m_r, m_{r+1}, ..., m_{r+t})$ where $m_{r+1} = ... = m_{r+t} = \infty$.

2 Calculations.

Let now Γ_1 be a subgroup of Γ_2 of finite index μ. Let v_i be the exponent of x_i modulo Γ_1, i.e. the least integer such that $x_i^{v_i} \in \Gamma_1$, (Here x_i denotes the generator of order m_i). It follows that $v_i < \infty$ and $v_i \mid m_i$ if $m < \infty$. Some of the x_i's in Γ_2 may have exponent m_i modulo Γ_1. Rearranging the periods so that $v_i = m_i$ only for $1 \leq i \leq p$ and x_{i+p} has exponent $n_i < m_{i+p}$ otherwise, we find that the signature of Γ_2 can be rewritten as $(g; m_1, ..., m_p, n_1 k_1, ..., n_q k_q)$ where $p + q = r + t$ and $1 < k_i \leq \infty$. Then Singerman's result can be deduced to the following form:

Theorem 1 *Let Γ_1 be a subgroup of Γ_2 of finite index μ. Then Γ_1 has signature*

$$\left(g_1; k_1^{\left(\frac{\mu}{n_1}\right)}, ..., k_q^{\left(\frac{\mu}{n_q}\right)} \right)$$

where $k_i^{\left(\frac{\mu}{n}\right)}$ means that the period k_i occurs $\frac{\mu}{n}$ times. Here g_1 can be found by the RHF.

By means of theorem 1, we can find the periods of both Λ and $\Theta^{-1}(\Lambda)$. Then it is easy to find g' in terms of g. In fact we obtain the following main result of this work:

Theorem 2 *Let Θ be the homomorphism between Γ and Δ, two discrete groups of the first kind, defined as above. Then the genus g of a subgroup Λ of Δ is equal to the genus g' of $\Theta^{-1}(\Lambda)$; i.e. Θ^{-1} preserves the genus.*

Proof. We prove this result in three cases. All other cases can be reduced to one of those. ■

Firstly, let

$$\Theta : \Gamma = (0; m_1, ..., m_r) \longrightarrow \Delta = (0; n_1, ..., n_r)$$

be a homomorphism for $m_i \geq 2$, $n_j \geq 1$, so that $n_i \mid m_i$ for every i, j. Let Λ be a subgroup of Δ of genus g. Then $\Theta^{-1}(\Lambda)$ is a subgroup of Γ with genus g, as well, i.e. Θ^{-1} preserves genus.

Let $\Theta^{-1}(\Lambda)$ have genus g', and let $y_1, ..., y_r$ be the generators of Γ. Then

$$\Theta(y_i) = (v_{i1})(v_{i2}) ... (v_{i\alpha})$$

such that for $1 \leq i \leq r$, where $k = [\Delta : \Lambda]$ and (v_{ij}) denotes a cycle of length v_{ij} in the permutation of $\Theta(y_i)$. Since Θ is an epimorphism, we also have $k = [\Gamma : \Theta^{-1}(\Lambda)]$. The periods of Λ are

$$\frac{n_1}{v_{11}}, ..., \frac{n_1}{v_{1\alpha_1}}, ..., \frac{n_r}{v_{r_1}}, ..., \frac{n_r}{v_{r\alpha_r}}$$

and the periods of $\Theta^{-1}(\Lambda)$ are

$$\frac{m_1}{v_{11}}, ..., \frac{m_1}{v_{1\alpha_1}}, ..., \frac{m_r}{v_{r_1}}, ..., \frac{m_r}{v_{r\alpha_r}}.$$

Hence by the Riemann-Hurwitz Formula,

$$2g-2+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{n_i}\right)=k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{n_i}\right)\right)$$

and

$$2g^{'}-2+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{m_i}\right)=k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right).$$

Now $g=g'$ if and only if

$$k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{n_i}\right)\right)-\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{n_i}\right)$$

$$=k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)-\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{m_i}\right)$$

if and only if

$$-k\sum_{i=1}^{r}\frac{1}{n_i}+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\frac{v_{ij}}{n_i}=-k\sum_{i=1}^{r}\frac{1}{m_i}+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\frac{v_{ij}}{m_i}$$

if and only if

$$-k\sum_{i=1}^{r}\frac{1}{n_i}+k\sum_{i=1}^{r}\frac{1}{n_i}=-k\sum_{i=1}^{r}\frac{1}{m_i}+k\sum_{i=1}^{r}\frac{1}{m_i}$$

since $\sum_{j=1}^{\alpha_i}v_{ij}=k$. Therefore for every k, $g=g\prime$.

Secondly, for $m_i\geq 2$, $n_j\geq 1$, so that for every i, $n_j\mid m_i$, let

$$\Theta:\Gamma=\left(0;m_1,...,m_r,\infty^{(t)}\right)\longrightarrow\Delta=\left(0;n_1,...,n_r,n_{r+1},...,n_{r+s},\infty^{(t-s)}\right)$$

be a homomorphism. Let Λ be a subgroup of Δ with genus g. Then $\Theta^{-1}(\Lambda)$ is a subgroup of Γ with genus g as well.

Let now $\Theta^{-1}(\Lambda)$ have genus g'. Then with the notation above

$$\Theta(y_i)=(v_{i1})...(v_{i\alpha}),\quad\text{for }1\leq i\leq r,$$

$$\Theta(y_{r+i})=(v_{r+i,1})...(v_{r+i,\alpha}),\quad\text{for }1\leq i\leq s,$$

$$\Theta\left(y_{r+i}\right)=\left(v_{r+i,1}\right)...\left(v_{r+i,\alpha}\right),\quad \text{for } s+1\leq i\leq t.$$

Then the periods of Λ are

$$\frac{n_1}{v_{11}},...,\frac{n_r}{v_{r\alpha_r}},\frac{n_{r+1}}{v_{r+1,1}},...,\frac{n_{r+1}}{v_{r+1,\alpha_{r+1}}},...,\frac{n_{r+s}}{v_{r+s,1}},...,\frac{n_{r+s}}{v_{r+s,\alpha_{r+s}}},...,\infty^{(\alpha_{r+s+1})},...,\infty^{(\alpha_{r+t})}$$

and the periods of $\Theta^{-1}(\Lambda)$ are

$$\frac{m_1}{v_{11}},...,\frac{m_1}{v_{1\alpha_1}},...,\frac{m_r}{v_{r1}},...,\frac{m_r}{v_{r\alpha_r}},...,\infty^{(\alpha_{r+1})},...,\infty^{(\alpha_{r+t})}.$$

Hence as above, we have $g=g'$ for every index k.

Thirdly and finally, $m_i\geq 2$, $n_j\geq 1$ and $n_j \mid m_i$ for $1\leq i\leq r$. Let

$$\Theta:\Gamma=\left(0;m_1,...,m_r,\infty^{(t)}\right)\longrightarrow\Delta=\left(0;n_1,...,n_{r+t}\right)$$

be a homomorphism. Let Λ be a normal subgroup of Δ of genus g. Then $\Theta^{-1}(\Lambda)$ is also of genus g.

Finally let g' be the genus of $\Theta^{-1}(\Lambda)$. First

$$\Theta\left(y_i\right)=\left(v_{i1}\right)...\left(v_{i\alpha}\right),$$

for $1\leq i\leq r+t$, such that

$$\sum_{j=1}^{\alpha_i}v_{ij}=k.$$

Then the periods of Λ are

$$\frac{n_1}{v_{11}},...,\frac{n_{1r}}{v_{1\alpha_1}},...,\frac{n_r}{v_{r1}},...\frac{n_r}{v_{r\alpha_r}},\frac{n_{r+1}}{v_{r+1,1}},...,\frac{n_{r+1}}{v_{r+1,\alpha_{r+1}}},...,\frac{n_{r+t}}{v_{r+t,1}},...,\frac{n_{r+t}}{v_{r+s,\alpha_{r+t}}},$$

and the periods of $\Theta^{-1}(\Lambda)$ are

$$\frac{m_1}{v_{11}},...,\frac{m_1}{v_{1\alpha_1}},...,\frac{m_r}{v_{r1}},...,\frac{m_r}{v_{r\alpha_r}},\infty^{(\alpha_{r+1})},...,\infty^{(\alpha_{r+t})}.$$

Again using the RHF, we obtain $g=g'$ for every index k.

Therefore we have completed the discussion of all three cases. All other cases can be reduced to one of these, e.g. if

$$\Theta:\Gamma=\left(0;m_1,...,m_r,\infty^{(t)}\right)\longrightarrow\Delta=\left(0;n_1,...,n_s,\infty^{(t)}\right),$$

this can be considered as a special case with $s=0$. If

$$\Theta:\left(0;\infty^{(t)}\right)\longrightarrow\left(0;n_1,...,n_s,\infty^{(t-s)}\right),0\leq s\leq t$$

this also can be considered as a special case with $r=0$. This completes the proof of Theorem 2.

Some applications of Theorem 2 has been done in the special case of Hecke groups $H(\lambda_q)$ in [1]. These are the discrete subgroups of $PSL(2,R)$ of the first kind having signature $(0;2,q,\infty)$ for $q\in\mathbb{Z}, q\geq 3$. Therefore they fall into the third class in our classification in the Introduction. In [1] and [3], a classification of some normal subgroups of $H(\lambda_q)$ has been done and this technique was often used to establish information about them.

As an example let us consider the homomorphism from $H(\lambda_q)$ to $(2,q,2q)$, for odd q, taking the generator R of order 2 of $H(\lambda_q)$ to the generator r of order 2, and the generator S of order q to the generator s order q. This is an infinite image of $H(\lambda_q)$, for odd q. If we take the subgroup Λ of $(2,q,2q)$ having the relations $r^2=s^q=rsrs^{-1}=1$, then Λ has the signature $\left(\frac{q-1}{2};\infty\right)$ by the RHF. Hence $\Theta^{-1}(\Lambda)$ has genus $\frac{q-1}{2}$ and also the same signature. Note that this gives us the commutator subgroup $\Theta^{-1}(\Lambda)=H'(\lambda_q)$ of $H(\lambda_q)$. When q is even, by mapping $H(\lambda_q)$ to $(2,q,2q)$ and applying the same method, we can obtain the commutator subgroup with signature $\left(\frac{q-1}{2}-1;\infty,\infty\right)$. Note that $H'(\lambda_q)$ is therefore isomorphic to a free group of rank $q-1$, [2].

References

[1] Cangul, I. N., *Normal Subgroups of Hecke Groups*, PhD Thesis, Southampton,1994.

[2] Cangul, I. N., *Determining Isomorphism Class of a Fuchsian Group from its Signature*, Academia Sinica, 29 (2001), 313-316.

[3] Cangul, I. N., Singerman, D., *Normal subgroups of Hecke Groups and Regular Maps*, Math. Proc, Camb. Phil. Soc., 123 (1998), 59-74.

[4] Singerman, D.T., *Subgroups of Fuchsian groups and Finite Permutation Groups*, Bull. LMS, 2 (1970), 319-323.

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