Rational curves in Grassmannians
and their Plücker embeddings

E. Ballico

Dept. of Mathematics, University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Fix integers \( k > n \geq 2 \) and \( a_1 \geq \cdots \geq a_n \) such that \( a_n \geq \lfloor \left( \binom{k}{n} - 1 \right) / n \rfloor \) and \( a_1 + \cdots + a_n + 1 \geq \binom{k}{n} \). Set \( E := \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i) \). Let \( V \) be a general \( k \)-dimensional linear subspace of \( H^0(\mathbb{P}^1, E) \). Here we prove that for all \( n \)-dimensional linear subspace \( W \) of \( V \) the evaluation map \( W \otimes \mathcal{O}_{\mathbb{P}^1} \to E \) is an injection of sheaves. Equivalently, the natural map \( \bigwedge^k(V) \to H^0(\mathbb{P}^1, \det(E)) \) is injective.

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1. Introduction

Theorem 1. Fix integers \( k > n \geq 2 \) and \( a_1 \geq \cdots \geq a_n \) such that \( a_n \geq \lfloor \left( \binom{k}{n} - 1 \right) / n \rfloor \) and \( a_1 + \cdots + a_n + 1 \geq \binom{k}{n} \). Set \( E := \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i) \). Let \( V \) be a general \( k \)-dimensional linear subspace of \( H^0(\mathbb{P}^1, E) \). Then for all \( n \)-dimensional linear subspace \( W \) of \( V \) the evaluation map \( W \otimes \mathcal{O}_{\mathbb{P}^1} \to E \) is an injection of sheaves. Equivalently, the natural map \( \bigwedge^k(V) \to H^0(\mathbb{P}^1, \det(E)) \) is injective.

Remark 1. The equivalence of the two statements appearing in Theorem 1 was pointed out by M. Teixidor i Bigas in [2]. Take \( E, V \) as in the statement of Theorem 1. Since \( E \) is spanned, \( k > n \), and \( V \) is general, \( V \) spans \( E \). Hence the pair \( (E, V) \) induces a morphism \( h_{E,V} : \mathbb{P}^1 \to G(n, k) \), where \( G(n, k) \) denote the Grassmannian of all \( (k - n) \)-dimensional linear subspaces of the vector space \( \mathbb{K}^n \). Let \( u_{n,k} : G(n, k) \to \mathbb{P}^{N(n,k)}, N(n, k) := \binom{n}{k} - 1, \) be the Plücker embedding. In [2] M. Teixidor i Bigas also proved that Theorem 1 is equivalent

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to show that \( u_{n,k} \circ h_{E,V}(\mathbb{P}^1) \) spans \( \mathbb{P}^{N(n,k)} \). We will prove Theorem 1 proving the non-degeneracy of the curve \( u_{n,k} \circ h_{E,V}(\mathbb{P}^1) \) inside \( \mathbb{P}^{N(n,k)} \).

For a smooth curve of genus \( g \geq 2 \) a result similar to Theorem 1 was proved in [1] when \( E \) is a general degree \( n \) stable vector bundle with sufficiently high degree. We stress that in Theorem 1 we do not require that \( E \) is rigid, i.e. we do not require the inequality \( a_1 \leq a_n - 1 \).

We work over an algebraically closed field \( \mathbb{K} \).

2. Proof of Theorem 1

For all integers \( b > a > 0 \) let \( G(a,b) \) denote the Grassmannian of all \((b-a)\)-dimensional linear subspaces of \( \mathbb{K}^b \). Thus \( \dim(G(a,b)) = a(b-a) \) and there is a tautological exact sequence of vector bundles on \( G(a,b) \)

\[
0 \to S_{G(a,b)} \to O_{G(a,b)}^{\oplus b} \to Q_{G(a,b)} \to 0
\]

with \( \text{rank}(Q_{G(a,b)}) = b \), \( \text{rank}(S_{G(a,b)}) = b-a \) and \( \det(Q_{G(a,b)}) \cong \det(S_{G(a,b)})^* \cong O_{G(a,b)}(1) \), where \( O_{G(a,b)}(1) \) denotes the positive generator of \( \text{Pic}(G(a,b) \cong \mathbb{Z} \). \( O_{G(a,b)}(1) \) is very ample and the associated complete linear system \( |O_{G(a,b)}(1)| \) induces the Plücker embedding \( u_{a,b} \) of \( G(a,b) \) into \( \mathbb{P}^{N(a,b)} \), \( N(a,b) := \binom{b}{a} - 1 \). We have \( TG(a,b) \cong Q_{G(a,b)} \otimes S_{G(a,b)}^* \). For all subscheme \( Z \) of \( G(a,b) \) let \( N_{Z,G(a,b)} \) denote its normal sheaf in \( G(a,b) \). Notice that \( N_{Z,G(a,b)} \) is a spanned vector bundle if \( Z \) is smooth. We will often see \( G(a,b) \) as the set of all \( \mathbb{P}^{a-1} \)'s contained in \( \mathbb{P}^{b-1} \). \( G(a,b) \) is a homogenous variety which contains many lines with respect to the Plücker embedding. Any such line is obtained in this way.

Fix an \( a \)-dimensional linear subspace \( B \) of \( \mathbb{P}^{b-1} \) and a codimension two linear subspace \( A \) of \( B \) (with the convention \( A = \emptyset \) if \( a = 1 \)). Set \( D(A,B) := \{ D \in G(a,b) : A \subset D \subset B \} \). \( D(A,B) \) is a line of \( G(a,b) \) and the group \( \text{Aut}(G(a,b)) \) acts transitively on the set \( \Gamma(a,b) \) of all lines contained in \( G(a,b) \). Take \( D \in \Gamma \).

The vector bundle \( Q_{S(a,b)}|D \) is a direct sum of one line bundle of degree 1 and \( a-1 \) line bundles of degree 0. The vector bundle \( S_{S(a,b)}|D \) is a direct sum of one line bundle of degree \( -1 \) and \( b-a-1 \) line bundles of degree 0. Thus \( TG(a,b)|D \) is a direct sum of one line bundle of degree 2, \( (a+b-2) \) line bundles of degree 1 and \( (a-1)(b-a-1) \) line bundles of degree 0. Thus \( N_{D,G(a,b)} \) is a direct sum of \( (a+b-2) \) line bundles of degree 1 and \( (a-1)(b-a-1) \) line bundles of degree 0.

The following lemma is well-known (see e.g. [1], §2).

**Lemma 1.** Let \( T \subset G(a,b) \) a reduced, connected and nodal curve such that \( p_a(T) = 0 \) and each irreducible component of \( T \) is a line. Then \( T \) is a smooth point of the Hilbert scheme \( \text{Hilb}(G(a,b)) \) of \( G(a,b) \). Furthermore, \( T \) is smoothable, i.e. it is the flat limit of a family of smooth and connected rational curves contained in \( G(a,b) \).

**Remark 2.** Fix integers \( b > a > 0 \) and \( d > 0 \). Let \( R(a,b,d) \) denote the set of all smooth and connected degree \( d \) rational curves contained in \( G(a,b) \). For
any \( Z \in R(a, b, d) \) the vector bundle \( N_{Z,G(a,b)} \) is a spanned vector bundle on \( Z \cong \mathbb{P}^1 \). Thus \( h^1(Z, N_{Z,G(a,b)}) = 0 \). Thus \( Z \) is a smooth point of the Hilbert scheme \( \text{Hilb}(G(a,b)) \). Furthermore, the vector bundles \( Q_{G(a,b)}|Z \) and \( S_{G(a,b)}|Z \) are rigid for any general element \( Z \) of any irreducible component of \( R(a, b, d) \) (use the universal properties of the Grassmannians \( G(a,b) \) and \( G(b-a, a) \) and that every vector bundle on \( \mathbb{P}^1 \) is a flat limit of a family of rigid vector bundles).

**Lemma 2.** Fix general \( P,Q \) in \( G(a,b) \). There is a chain of \( b-a \) lines joining \( Q \) with \( P \), i.e. \( b-a \) lines \( D_1, \ldots, D_{b-a} \) such that \( P \in D_1, Q \in D_{b-a}, D_i \cap D_{i+1} \neq \emptyset \) for all \( i \in \{1, \ldots, b-a-1\} \), \( T := D_1 \cup \cdots \cup D_{b-a} \) is connected and nodal and \( p_a(T) = 0 \).

**Proof.** Since \( G(a,a+1) \cong \mathbb{P}^a \), the case \( b = a+1 \) is trivial. Use induction on \( b \) and the description of all lines contained in \( G(a,b) \).

**Lemma 3.** Fix integers \( b > a > 0 \). Then there exists a reduced, connected and nodal curve such that \( \deg(T) = \binom{b}{a} - 1, p_a(T) = 0 \), each irreducible component of \( T \) is a line and \( u_{a,b}(T) \) spans \( \mathbb{P}^{N(a,b)} \).

**Proof.** Since \( G(a,a+1) \cong \mathbb{P}^a \), the case \( b = a+1 \) is trivial. Use induction on \( b \) and Lemma 2.

**Proof of Theorem 1.** Set \( x := \lfloor (\binom{k}{n} - 1)/n \rfloor \) and \( y := (\binom{k}{n}) - 1 - nx \). Let \( F \) be the rank \( n \) vector bundle on \( \mathbb{P}^1 \) isomorphic to the direct sum of \( y \) line bundles of degree \( x+1 \) and \( n-y \) line bundles of degree \( x \), i.e. the rigid line bundle with rank \( n \) and degree \( \binom{k}{n} \). Notice that \( h^0(\mathbb{P}^1, F) = \binom{k}{n} + n - 1 \geq k \).

Our assumptions on \( a_n \) and \( a_1 + \cdots + a_n \) imply the existence of an inclusion of sheaves \( j : F \rightarrow E \). The map \( j \) induces an inclusion \( j_* : H^0(\mathbb{P}^1, F) \rightarrow H^0(\mathbb{P}^1, E) \). By semicontinuity it is sufficient to prove the result for one \( k \)-dimensional linear subspace, e.g. one of the form \( j_*(M) \) with \( M \) a general \( k \)-dimensional linear subspace of \( H^0(\mathbb{P}^1, F) \). Apply Lemmas 1 and 3, Remark 2 and the interpretation of Theorem 1 given in the last part of Remark 1.

**References**


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