A note on moment inequality for harmonic used better than aged in expectation (HUBAE) class of life distributions with hypothesis testing application

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Abstract

The harmonic used better than aged in expectation HUBAE class of life distributions is considered. A moment inequality is derived for HUBAE distributions which demonstrate that if the mean life is finite, then all moments exist. Based on this inequality, a new test statistic for testing exponentiality against HUBAE is introduced. It is shown that the proposed test is simple, has high relative efficiency for some commonly used alternatives. Critical values are tabulated for sample sizes $n = 5(1)40$. The set of real data is used as a practical application of the proposed test in the medical science.

Keywords: HUBAE, exponentiality, efficiency, moments, asymptotic normality.
1 Introduction

Let $X$ be a nonnegative continuous random variable with distribution function $F(x)$, survival function $S(t) = 1 - F(t)$, at age $t$, we define the random residual life by $X_t$ with survival function $S(t) = F(t+x) / F(t)$, $x, t \geq 0$ and assume that $X$ has a finite mean $\mu = E(X) = \int_0^\infty F(u)du$. Some properties concerning the asymptotic behavior of $X_t$ as $t \to \infty$ will be used. Bhattacharjee(1982) gave the following definition.

**Definition(1.1).** If $X$ is nonnegative random variable, its distribution function $F$ is said to be finitely and positively smooth if a number $\gamma \in (0, \infty)$ exists such that:

$$
\lim_{t \to \infty} \frac{F(t+x)}{F(t)} = e^{-x\gamma},
$$

(1.1)

Where $\gamma$ called the asymptotic decay coefficient of $X$. Denoting $X_e$ be a random variable exponentially distributed by mean $1/\gamma$, the following definitions imply that $X_t$ converges to $X_e$ in distribution written as $X_t \xrightarrow{d} X_e$. This property is useful for discretioin of random life times of devices of unknown age.

**Definition(1.2).** The distribution $F$ is said to be used better than age UBA if for all $x, t \geq 0$

$$
\int_0^\infty F(x+t)dx \geq F(t)e^{-\gamma x},
$$

(1.2)

where $\gamma$ called is the asymptotic decay of $X$. From definition (1.2) we have the following definition:

**Definition(1.3).**The distribution $F$ is said to be harmonic used better than aged in expectation HUBAE if for all $x, t \geq 0$

$$
\int_x^\infty F(t)dt \geq \mu e^{-\gamma x},
$$

(1.3)

where $\gamma$ is asymptotic decay of $X$.

We observe that the inequality of (1.3) is achieved when $F(x)$ has an exponential distribution with mean $\mu$ equal to the coefficient of the asymptotic decay $\gamma$, where the exponential distribution is the only distribution which has the lack of memory property.

Alzaid (1994) showed that UBA class of life distribution is a subclass of the used better than age in expectation class (UBA) and that if $F$ is IHR (increasing hazard rate), then $F$ is UBA. Similar implications between UBAE,
NBUE and HNBUE were given by Di Crescenzo (1999). More recently, Willmot and Cai (2000) showed that the UBA class includes the DMRL class (decreasing mean residual life) while the UBAE includes the DVRL (decreasing variance residual life). Thus we have

\[ \text{IHR} \subset \text{DMRL} \subset \text{UBA} \subset \text{UBAE} \subset \text{HUBAE} \cup \text{DVRL} \]

For definitions and details of the classes IHR, NBUE and DMRL, see Barlow and Proschan (1981) and for HNBUE see Klefsjo (1983) while for DVRL see Abu-Youssef (2004).

Several authors derived moments inequalities of different families of life distributions such as IHR, IHRA, NBU, NBUE, HNBUE, HRNBU, HRNBUE, NRBUE, UBA and UBAE, cf. Ahmad (2001, 2004), Ahmad and Mugadi (2004) and Abu-Youssef (2003, 2004). Testing exponentiality against the classes of life distribution has seen a good deal of attention. For testing against IHR, we refer to Barlow and Proschan (1981) and Ahmad (1994), among others. While testing against DMRL see Ahmad (1992) and testing against DVRl see Abu-yossef (2004). Finally testing against UBA and UBAE see Ahmad (2004).

The thread that connects most work mentioned here is that a measure of departure from \( H_0 \), which is often some weighted function of \( F \), is developed which is strictly positive under \( H_1 \) and is zero under \( H_0 \). Then, a sample version of this measure is used as test statistics and its properties are studied. In the present work, the moment inequality developed in section 2 can be used to construct test statistic for HUBAE. In section 3 this test statistic is based on sample moments of aging distribution. This test statistic is simple to drive, and has exponentially high efficiencies for the well known alternatives relative to other tests. Montecarlo null distribution critical points obtained for sample sizes 5(1)40. Finally we apply the proposed test to real practical data in medical science given in Aboummah et al. (1994).

### 2 Moment Inequality

We state and prove the following theorem.

**Theorem 2.1.** If \( F \) is HUBAE, then

\[
\gamma^{r+1} \mu_{(r+2)} \geq (r+2)! \mu, \quad r \geq 0. \tag{2.1}
\]
where
\[ \mu_{(r+2)} = (r + 2) \int_0^\infty x^{r+1} \overline{F}(x) \, dx. \]

**Proof.** Since \( F \) is HUBAE, then
\[ \nu(x) \geq \mu e^{-x\gamma}, \quad (2.2) \]
where
\[ \nu(x) = \int_x^\infty \overline{F}(t) \, dt. \]

Multiplying both sides by \( x^r, \ r \geq 0 \), and integrating over \((0, \infty)\), w.r.t. \( x \), then
\[ \int_0^\infty x^r \nu(x) \, dx \geq \mu \int_0^\infty x^r e^{-x\gamma} \, dx \quad (2.3) \]
the left hand side of (2.3) is
\[ \int_0^\infty x^r \nu(x) \, dx = -\frac{1}{r + 1} \int_0^\infty x^{r+1} \nu'(x) \, dx. \tag{2.4} \]

Since \( \nu'(x) = -\overline{f}(x) \), then
\[ \int_0^\infty x^r \nu(x) \, dx = \frac{1}{r + 1} \int_0^\infty x^{r+1} \overline{F}(x) \, dx = \frac{\mu_{r+2}}{(r + 2)(r + 1)}. \tag{2.5} \]
The right hand side of (2.3) is given by
\[ \int_0^\infty \mu e^{-x\gamma} x^r \, dx = \frac{\mu r!}{\gamma^{r+1}}. \tag{2.6} \]
By using (2.5) and (2.6), (2.1) is obtained.

3 Applications to hypotheses testing

3.1 Testing against HUBAE alternatives

Let \( X_1, X_2, \ldots, X_n \) represent a random sample from a population with distribution \( F \). We wish to test the null hypothesis \( H_0 : \overline{F} \) is exponential with mean \( \mu \) against \( H_1 : \overline{F} \) is HUBAE and not exponential. Using theorem (2.1), we may use the following \( \delta_h \) as a measure of departure from \( H_0 \) infavor of \( H_1 \):
\[ \delta_h = \gamma^{r+1} \mu_{(r+2)} - (r + 2)! \mu \tag{3.1} \]
Note that under \( H_0 : \delta_h = 0 \), while under \( H_1 : \delta_h > 0 \). Thus to estimate \( \delta_h \) by \( \hat{\delta}_h \), let \( X_1, X_2, \ldots, X_n \) be a random sample from \( F \), \( \hat{\gamma} = \frac{n}{\sum X_i} \) is the estimate
of $\gamma$ and $\mu$ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n} \sum X_i$ is the usual sample mean. Then $\hat{\delta}_h$ is given by using (3.1) as

$$\hat{\delta}_h = \frac{1}{n} \sum_i \left\{ \gamma^{r+1} X_i^{r+2} - (r + 2)! X_i \right\}.$$ \hspace{1cm} (3.2)

to make the test statistic scale invariant, we use

$$\Delta_h = \frac{\hat{\delta}_h \mu}{\mu}.$$  

which is estimated by

$$\hat{\Delta}_h = \frac{\hat{\delta}_h X}{X}.$$ \hspace{1cm} (3.3)

Setting $\phi(X_1) = \gamma^{r+1} X_1^{r+2} - (r + 2)! X_1$, then $\hat{\Delta}_h$ in (3.3) is a U-statistic, cf. Lee (1990). The following theorem summarizes the large sample properties of $\hat{\Delta}_h$.

**Theorem 3.1.** As $n \to \infty$, $\sqrt{n}(\hat{\Delta}_h - \Delta_h)$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = \text{var}[\gamma^{r+1} X_1^{r+2} - (r + 2)! X_1]$$ \hspace{1cm} (3.4)

Under $H_0 : \Delta_h = 0$ and variance $\sigma_0^2$ is given by

$$\sigma_0^2 = (2r + 4)! + 2((r + 2)!)^2 X^2 - 2(r + 2)(r + 3)!$$ \hspace{1cm} (3.5)

**Proof:** Since $\hat{\Delta}_h$ and $\hat{\delta}_h \mu / \mu$ have the same limiting distribution, we use $\sqrt{n}(\hat{\delta}_h - \delta_h)$. Now this is asymptotically normal with mean 0 and variance $\sigma^2 = \text{var}[\gamma^{r+1} \phi(X_1)]$, where

$$\phi(X_1) = X_1^{r+2} - (r + 2)! X_1.$$  

Then (3.4) follows.

Under $H_0 \Delta_h = E(\phi(X_1)) = 0$ and

$$\sigma_0^2 = E[X_1^{r+2} - (r + 2)! X]^2.$$ \hspace{1cm} (3.6)

Hence (3.5) follows. The Theorem is proved.

When $r = 0$, \hspace{1cm} $\delta_h = \mu(\gamma) - 2\mu,$ \hspace{1cm} (3.7)
in this case \( \sigma_0^2 = 8 \) and the test statistic

\[
\hat{\delta}_h = \frac{1}{n} \sum \left\{ X_i^2 \hat{\gamma} - 2X_i \right\}
\]

(3.8)

and

\[
\hat{\Delta}_h = \frac{\hat{\delta}_h}{X^2},
\]

(3.9)

which is quite simple statistics.

To use the above test, calculate \( \sqrt{n} \hat{\Delta}_h / \sigma_0 \) and reject \( H_0 \) if this exceeds the normal variate value \( Z_{1-\alpha} \). To illustrate the test, we calculate, via Monte Carlo Method, the empirical critical points of \( \hat{\Delta}_h \) in (3.9) for sample sizes 5(1)40. Tables (3.1) gives the upper percentile points for 95%, 98%, 99% . The calculations are based on 5000 simulated samples sizes \( n = 5(1)40 \).
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Table (3.1) Critical Values of $\hat{\Delta}_h_n$ in (3.9)

<table>
<thead>
<tr>
<th>$n$</th>
<th>95%</th>
<th>98%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5486</td>
<td>0.9004</td>
<td>1.1503</td>
</tr>
<tr>
<td>6</td>
<td>0.5779</td>
<td>1.0236</td>
<td>1.2910</td>
</tr>
<tr>
<td>7</td>
<td>0.5940</td>
<td>0.9426</td>
<td>1.2027</td>
</tr>
<tr>
<td>8</td>
<td>0.6425</td>
<td>1.0617</td>
<td>1.3685</td>
</tr>
<tr>
<td>9</td>
<td>0.6168</td>
<td>1.0118</td>
<td>1.2873</td>
</tr>
<tr>
<td>10</td>
<td>0.6263</td>
<td>1.0513</td>
<td>1.3720</td>
</tr>
<tr>
<td>11</td>
<td>0.6315</td>
<td>1.0132</td>
<td>1.3312</td>
</tr>
<tr>
<td>12</td>
<td>0.6598</td>
<td>1.0639</td>
<td>1.3395</td>
</tr>
<tr>
<td>13</td>
<td>0.6350</td>
<td>0.9947</td>
<td>1.2981</td>
</tr>
<tr>
<td>14</td>
<td>0.5991</td>
<td>0.9572</td>
<td>1.1516</td>
</tr>
<tr>
<td>15</td>
<td>0.6061</td>
<td>1.0030</td>
<td>1.2714</td>
</tr>
<tr>
<td>16</td>
<td>0.6469</td>
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<td>17</td>
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</tr>
<tr>
<td>19</td>
<td>0.5709</td>
<td>0.9030</td>
<td>1.0829</td>
</tr>
<tr>
<td>20</td>
<td>0.5989</td>
<td>0.9197</td>
<td>1.1594</td>
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<tr>
<td>21</td>
<td>0.5475</td>
<td>0.8552</td>
<td>1.1012</td>
</tr>
<tr>
<td>22</td>
<td>0.5837</td>
<td>0.8623</td>
<td>1.0519</td>
</tr>
<tr>
<td>23</td>
<td>0.5838</td>
<td>0.8949</td>
<td>1.1339</td>
</tr>
<tr>
<td>24</td>
<td>0.5638</td>
<td>0.8391</td>
<td>1.0430</td>
</tr>
<tr>
<td>25</td>
<td>0.5666</td>
<td>0.7838</td>
<td>0.9764</td>
</tr>
<tr>
<td>26</td>
<td>0.5393</td>
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<td>1.0541</td>
</tr>
<tr>
<td>27</td>
<td>0.5472</td>
<td>0.8482</td>
<td>1.1541</td>
</tr>
<tr>
<td>28</td>
<td>0.5315</td>
<td>0.8060</td>
<td>0.9822</td>
</tr>
<tr>
<td>29</td>
<td>0.5252</td>
<td>0.8342</td>
<td>0.9951</td>
</tr>
<tr>
<td>30</td>
<td>0.5370</td>
<td>0.7905</td>
<td>0.9528</td>
</tr>
<tr>
<td>31</td>
<td>0.5101</td>
<td>0.7572</td>
<td>1.0231</td>
</tr>
<tr>
<td>32</td>
<td>0.5068</td>
<td>0.7516</td>
<td>0.9644</td>
</tr>
<tr>
<td>33</td>
<td>0.5087</td>
<td>0.7564</td>
<td>0.9648</td>
</tr>
<tr>
<td>34</td>
<td>0.5083</td>
<td>0.6984</td>
<td>0.8729</td>
</tr>
<tr>
<td>35</td>
<td>0.5216</td>
<td>0.7728</td>
<td>0.9945</td>
</tr>
<tr>
<td>36</td>
<td>0.4832</td>
<td>0.7416</td>
<td>0.9544</td>
</tr>
<tr>
<td>37</td>
<td>0.5061</td>
<td>0.7201</td>
<td>0.8976</td>
</tr>
<tr>
<td>38</td>
<td>0.4809</td>
<td>0.7142</td>
<td>0.8965</td>
</tr>
<tr>
<td>39</td>
<td>0.4758</td>
<td>0.7309</td>
<td>0.8848</td>
</tr>
<tr>
<td>40</td>
<td>0.4847</td>
<td>0.6692</td>
<td>0.8451</td>
</tr>
</tbody>
</table>

To assess how good this procedure is relative to others in the literatures, we use the concept of Pitman’s asymptotic efficiency (PAE). To do this we need
to evaluate PAE of the proposed test and compare it with other tests. Since the above test statistic $\hat{\Delta}_{hn}$ in (3.3) is new and no other tests are known for these class HUBAE. We may compare it with smaller classes such as (DMRL), and UBAE. Here we choose the tests $K^*$ and $\hat{\delta}_2$ were presented by Hollander and Prochan (1975) and Ahmad (2004) respectively for decreasing mean residual life class (DMRL) and used better than aged in expectation (UBAE) class. Note that PAE of $\hat{\Delta}_{hn}$ is given by

$$
PAE(\Delta_h(\theta)) = \frac{d}{d\theta}E(\theta)|_{\theta=\theta_0} / \sigma_0.
$$

(3.10)

Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i) Linear failure rate family : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$, $x > 0, \theta > 0$

(ii) Makeham family : $\bar{F}_{2\theta} = e^{-x-\theta(x+e^{-x}-1)}$, $x > 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makeham families.

The PAE’s of these alternatives of our procedure are, respectively:

$$
PAE(\Delta_h, LFR) = -\frac{(r + 2)(r + 3)!}{2} + (r + 2)!,
$$

$r \geq 0$

(3.11)

$$
PAE(\Delta_h, Makeham) = -(r + 2)(r + 1)![r + 1 + \frac{1}{2r+2}] + \frac{(r + 2)!}{2}
$$

(3.12)

Direct calculations of PAE of $K^*$, $\hat{\Delta}_2$ and $\hat{\Delta}_{hn}$ are summarized in Table (3.2).

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$K^*$</th>
<th>$\hat{\delta}_2$</th>
<th>$\hat{\Delta}_{hn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$ Linear failure rate</td>
<td>0.806</td>
<td>0.630</td>
<td>1.41</td>
</tr>
<tr>
<td>$F_2$ Makeham</td>
<td>0.289</td>
<td>0.385</td>
<td>0.53</td>
</tr>
</tbody>
</table>

From Table (3.2), the test statistic $\hat{\Delta}_{hn}$ is more efficient than $\hat{\Delta}_2$ and $K^*$ for linear failure rate family, and Makeham family. Note that: Since $\hat{\Delta}_{hn}$ defines a class (with parameter) $r$ of test statistic, we choose $r$ that the maximizes the PAE of that alternatives. If we take $r = 1$ then our test will have more efficiency than others.
4 Numerical Examples

Consider the data in Abouammoh et al (1994). These data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospital in Saudi Arabia and the ordered life times (in day) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852.

Using equation (3.9), the value of test statistics, based on the above data is $\hat{\Delta}_h = -0.0072$. This value leads to the acceptance of $H_0$ at the significance level $\alpha = 0.95$ see Table (3.1). Therefore the data has't HUBAE Property.

References


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