On the Number of Minimal Dominating Sets in Some Classes of Trees

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Abstract
A subset \( Q \subseteq V(G) \) is a dominating set of a graph \( G \) if each vertex in \( V(G) \) is either in \( Q \) or is adjacent to a vertex in \( Q \). A dominating set \( Q \) of \( G \) is minimal if \( Q \) contains no dominating set of \( G \) as a proper subset. In this paper we study the number of minimal dominating sets in some classes of trees.

Mathematics Subject Classification: 05C69

Keywords: dominating sets; minimal dominating sets; counting

1 Introduction

In general we use the standard terminology and notation of graph theory, see [2]. Let \( G \) be a simple, undirected graph. \( P_n \) denotes a path on \( n \) vertices, \( n \geq 0 \). Let \( X \subset V(G) \cup E(G) \). The notation \( G \setminus X \) means the graph obtained from \( G \) by deleting the set \( X \). A subset \( Q \subseteq V(G) \) is a dominating set of \( G \) if any vertex \( x \in V(G) \setminus Q \) is adjacent to at least one vertex \( y \in Q \). A minimal dominating set \( Q \) of \( G \) is a dominating set that contains no dominating set of \( G \) as a proper subset. Throughout this paper for convenience we will write a md-set of \( G \) instead of a minimal dominating set of \( G \). By \( \text{NMD}(G) \) we will denote the total number of md-sets in \( G \).

The concept of domination in graphs has existed in the literature for a long time, see [1, 3, 4]. In this paper we study the number of all md-sets in trees. We count md-sets in special classes of trees.

Let \( x \in V(G) \). We denote the family of all md-sets \( Q \) of \( G \) such that \( x \in Q \) (respectively \( x \not\in Q \)) by \( Q_x \) (respectively \( Q_{\neg x} \)). Then the basic rule for
counting md-sets in a graph $G$ is as follows

$$\text{NMD}(G) = |Q_x| + |Q_{\sim x}|.$$ 

In what follows $T$ stands for a tree with vertex set $V(T)$. Recall that a vertex of degree 1 is called a leaf. For $x \in V(T)$ the set of all leaves attached to the vertex $x$ is denoted by $L(x)$. From the definition of md-set immediately follows:

**Proposition 1** Let $T$ be a tree with $x \in V(T)$. Assume that $L(x) = \{z_1, \ldots, z_k\}$, $k \geq 2$, is the set of leaves attached to the vertex $x$ and $L'(x)$ is an arbitrary proper subset of $L(x)$. Then $\text{NMD}(T) = \text{NMD}(T \setminus L'(x))$. \qed

## 2 Main results

It is clear that if $|V(T)| = 1$ then $\text{NMD}(T) = 1$. If $|V(T)| = 2$ then $\text{NMD}(T) = 2$. We consider $n$-vertex trees with $n \geq 3$.

**Theorem 2** Let $T$ be an $n$-vertex tree with $n \geq 3$. Then $\text{NMD}(T) = 2$ with equality if and only if $T = K_{1,n-1}$.

**Proof.** It is clear that $\text{NMD}(K_{1,n-1}) = 2$. Assume that $T$ is an $n$-vertex tree with $n \geq 3$ and $\text{NMD}(T) = 2$. We shall show that $T = K_{1,n-1}$. If $n = 3$ then $T = K_{1,2}$. Assume that $n \geq 4$ and $T \neq K_{1,n-1}$. This means that there exists a path of length at least 3 in $T$. Let $uxyz$ be a path in $T$ with $u$ being a leaf. Of course, $\text{NMD}(T) = |Q_u| + |Q_{\sim u}|$. Let $Q \in Q_u$ be a md-set of $T$. Since $u \in Q$, we see that $x \notin Q$. Because $y$ is not a leaf, we have possibilities that $y \in Q$ or $y \notin Q$. Hence there are at least two different md-sets in $T$ containing $u$, so $|Q_u| \geq 2$. Assume now that $Q' \in Q_{\sim u}$ is a md-set of $T$. Of course $x \notin Q'$. Analogously as above, $y \in Q'$ or $y \notin Q'$, so $|Q_{\sim u}| \geq 2$. Consequently, we obtain that $\text{NMD}(T) = |Q_u| + |Q_{\sim u}| \geq 4$. This contradicts our assumption. \qed

**Theorem 3** For an arbitrary $n \geq 3$ there is no an $n$-vertex tree $T$ with $\text{NMD}(T) = 3$.

**Proof.** If $n = 3$ then the theorem is clear. Assume that $n \geq 4$ and there is an $n$-vertex tree $T$ with $\text{NMD}(T) = 3$. Theorem 2 shows that $T \neq K_{1,n-1}$. Hence there is a path in $T$ of length at least 3. As in Theorem 2 we obtain that $\text{NMD}(T) \geq 4$, a contradiction. \qed

Let $K_{1,p_i}$ be a star with $p_i \geq 2$ for $i = 1, \ldots, n$. We define a class of graphs denoted by $S_{p_1, \ldots, p_m}^n$, $p_i \geq 2$, $i = 1, \ldots, m$, with $n$ vertices, $n = \sum_{i=1}^m p_i + m$, obtained recursively from the graph $S_{p_1, \ldots, p_{m-1}}^{m-1}$ by joining the center of $(p_m+1)$-vertex star to a non-leaf of $S_{p_1, \ldots, p_{m-1}}^{m-1}$. Moreover $S_{p_1}^1 = K_{1,p_1}$.
Theorem 4 Let \( n \geq 3, \ p_i \geq 2 \) for \( i = 1, \ldots, m \) be integers. Then

\[
\text{NMD}(S_{p_1, \ldots, p_m}^m) = 2 \text{NMD}(S_{p_1, \ldots, p_m}^{m-1})
\]

with the initial condition \( \text{NMD}(S^1_{p_1}) = 2 \).

**Proof.** The initial condition follows from the Theorem 2. Let \( m \geq 2 \). Assume that \( x_k \) is not a leaf in \( S_{p_1, \ldots, p_m}^m \) and \( L(x_k) = y_1, \ldots, y_t, t \geq 2 \). By general rule of counting we have \( \text{NMD}(S_{p_1, \ldots, p_m}^m) = |Q_{x_k}| + |Q_{-x_k}| \). Let \( Q \in Q_{x_k} \). Then \( x_k \in Q \) and \( y_i \notin Q \) for every \( i = 1, \ldots, t \). Hence \( Q = \{x_k\} \cup Q' \) where \( Q' \) is an arbitrary md-set of the graph \( S_{p_1, \ldots, p_m}^m \setminus \{x_k, y_1, \ldots, y_t\} \) isomorphic to \( S_{p_1, \ldots, p_m}^{m-1} \). This implies that \( |Q_{x_k}| = \text{NMD}(S_{p_1, \ldots, p_m}^{m-1}) \). Let now \( Q \in Q_{-x_k} \). Then \( x_k \notin Q \) and \( y_i \in Q \) for \( i = 1, \ldots, t \). Hence \( Q = \{y_i; i = 1, \ldots, t\} \cup Q' \) where \( Q' \) is defined as above. Consequently \( |Q_{-x_k}| = \text{NMD}(S_{p_1, \ldots, p_m}^{m-1}) \). Finally, \( \text{NMD}(S_{p_1, \ldots, p_m}^m) = 2 \text{NMD}(S_{p_1, \ldots, p_m}^{m-1}) \) which ends the proof.

Corollary 5 Let \( n \geq 3, \ p_i \geq 2 \) for \( i = 1, \ldots, m \) be integers. Then

\( \text{NMD}(S_{p_1, \ldots, p_m}^m) = 2^m \). Moreover, \( \text{NMD}(S_{p_1, \ldots, p_m}^m) \) has the maximum value if \( p_i = 2 \) for every \( i = 1, \ldots, m \).

Theorem 6 Let \( T \) be an \( n \)-vertex tree, \( n \geq 3 \). Then for an arbitrary \( m > n \) there is an \( m \)-vertex tree \( T^* \) such that \( \text{NMD}(T^*) = \text{NMD}(T) \).

**Proof.** Let \( x \in V(T) \) and \( L(x) \neq \emptyset \). We locally augment the tree \( T \) by adding to the vertex \( x \) the star \( K_{1,p} \), \( p = m - n \), so that the vertex \( x \) is identified with the center \( y \) of the star \( K_{1,p} \). Then we obtain the tree \( T^* \) with \( |V(T^*)| = m \) and \( \text{NMD}(T^*) = \text{NMD}(T) \), which ends the proof.

Proposition 7 Let \( n \geq 1 \) be an integer. Then for \( n \geq 6 \)

\[
\text{NMD}(P_n) = \text{NMD}(P_{n-2}) + \text{NMD}(P_{n-3}) + \text{NMD}(P_{n-4}) - \text{NMD}(P_{n-6})
\]

with initial conditions \( \text{NMD}(P_0) = \text{NMD}(P_1) = 1, \text{NMD}(P_2) = \text{NMD}(P_3) = 2, \text{NMD}(P_4) = \text{NMD}(P_5) = 4 \).

**Proof.** Assume that vertices of \( P_n \) are numbered in the natural fashion. The initial conditions are obvious. Assume that \( n \geq 7 \). It is clear that \( \text{NMD}(P_n) = |Q_{x_n}| + |Q_{-x_n}| \). Assume that \( Q \) is an arbitrary md-set of \( T \). If \( Q \in Q_{x_n} \) then \( x_n \in Q \). It is easily seen that \( x_{n-1} \notin Q \). This implies that \( Q = Q' \cup \{x_n\} \) where \( Q' \) is an arbitrary md-set of the graph \( P_n \setminus \{x_n, x_{n-1}\} \), which is isomorphic to \( P_{n-2} \). Hence \( |Q_{x_n}| = \text{NMD}(P_{n-2}) \). Let now \( Q \in Q_{-x_n} \). This means that \( x_n \notin Q \) and \( x_{n-1} \in Q \). Let \( f(n), n \geq 4, \) be the total number of md-sets of \( P_n \) containing vertices \( x_{n-1}, x_{n-2} \). Consider the following cases.
(1) \( x_{n-2} \notin Q \).
Let \( Q' \subseteq Q_{-n} \) be a subfamily of md-sets such that \( x_{n-2} \notin Q \) for all \( Q \in Q' \). Then \( Q = Q'' \cup \{x_{n-1}\} \), where \( Q'' \) is an arbitrary md-set of the graph \( P_n \setminus \{x_n, x_{n-1}, x_{n-2}\} \) which is isomorphic to \( P_{n-3} \). Hence, \( |Q'| = NMD(P_{n-3}) \).

(2) \( x_{n-2} \in Q \).
Let \( Q'' \subseteq Q_{-x_n} \) be a subfamily of md-sets such that \( x_{n-2} \notin Q \) for all \( Q \in Q'' \). Then \( x_{n-3}, x_{n-4} \notin Q \), and by definition, \( |Q''| = f(n) \). Consider the following subcases.

(2.1) \( x_{n-5} \in Q \) and \( x_{n-6} \notin Q \).
Then it is clear that \( Q = Q_1 \cup \{x_{n-5}, x_{n-2}, x_{n-1}\} \), where \( Q_1 \) is an arbitrary md-set of the graph \( P_n \setminus \{x_{n-i}; i = 0, \ldots, 6\} \), which is isomorphic to \( P_{n-7} \). Hence we have \( NMD(P_{n-7}) \) sets \( Q \) in this case.

(2.2) \( x_{n-5} \in Q \) and \( x_{n-6} \in Q \).
In this case \( Q = Q_2 \cup \{x_{n-1}, x_{n-2}\} \), where \( Q_2 \) is an arbitrary md-set of the graph \( P_n \setminus \{x_{n-j}; j = 0, \ldots, 3\} \), which is isomorphic to \( P_{n-4} \), and \( Q_2 \) contains vertices \( x_{n-5}, x_{n-6} \). Therefore, due to the definition of the number \( f(n) \), we have \( f(n-4) \) sets \( Q \) in this case. Consequently, from the above possibilities we obtain that \( f(n) = NMD(P_{n-7}) + f(n-4), n \geq 7 \).

Finally for \( n \geq 7 \), \( NMD(P_n) = NMD(P_{n-2}) + NMD(P_{n-3}) + f(n) \) and \( f(n) = NMD(P_{n-7}) + f(n-4) \). Hence we obtain for \( n \geq 6 \)
\( NMD(P_n) = NMD(P_{n-2}) + NMD(P_{n-3}) + NMD(P_{n-4}) - NMD(P_{n-6}) \). \( \square \)

References


Received: September, 2010