On Noetherian Regular $\delta$-Near Rings and their Extensions

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Abstract

A commutative ring $N$ is said to be a Noetherian Regular $\delta$-Near Ring if every prime ideal of $N$ is strongly prime. We say that a commutative Noetherian Near Ring $N$ is Noetherian Regular Near Ring if $N$ is a Noetherian Regular $\delta$-Near Ring if assasinator of every right ideal (i.e., a right $N$-module) is strongly Prime Ideal. In this paper we obtained some fundamental results related to Noetherian Regular $\delta$-Near Ring and semi-Noetherian Regular $\delta$-Near Rings.

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1 Introduction

Let us recall that a prime ideal $P$ of a ring $N$ is said to be divided if it is comparable under set inclusion to every ideal of $N$. A $\delta$-Near Ring $N$ is called...
a "Regular $\delta$-Near Ring" if a sub-direct product of subdirectly irreducible $\delta$-Near Ring $N_i$ is isomorphic to a $\delta$-Near Ring $N$.

Since, each $N_i$ is isomorphic image of $N$ $\delta$-Near Ring and $N$ has the IFP follows then $N$ is a Regular $\delta$-Near Ring. Let $N$ be a semi prime commutative Noetherian $Q$-Algebra, $\phi$ be an automorphism of $N$ such that $N$ is a $\phi (x)$ ring and $\delta$ is $\delta$-Near-Ring then

(i) if for any $U \in S.spec(N)$ with $\phi(U) = U$ and $\delta(U) \subseteq U \Rightarrow o(U) \in S.spec(N)$, then $N$ is a semi-Noetherian Regular $\delta$-Near Ring $\Rightarrow N(x; \phi, \delta)$ is a semi-Noetherian Regular $\delta$-Near-Ring.

(ii) if $N$ is a semi-Noetherian Regular Near-Ring then $o(N)$ is a Noetherian Regular $\delta$-Near- Ring.

Throughout this paper, by a near-rings we mean zero symmetric near-ring for the basic terminology and notation the reader referred to Gunter Pilz [3]. The concept of Noetherian near-rings and Noetherian $d$-Near Rings was studied by by S. Ligh [5], Y.V.Reddy, C.V.L.N.Murthy. [7] and some others. In this paper we studied the concepts of Boolean Regular near-rings and Boolean Regular $\delta$-Near Ring and obtained some results related to these concepts.

All rings are Associative with identity throughout this paper $N$ denotes a commutative Ring with identity $1 \neq 0$.

The set of all nilpotent elements of $N$ and the Prime radical of $N$ are denoted by $N(N)$ and $P(N)$ respectively. The field of rational numbers and the ring of integers are denoted by $Q$ and $Z$ respectively. Unless and otherwise stated. For any subset $J$ of $N$, right $N$-Module $M$, Annihilator Ann. Of $J$ is denoted by Ann.$(J)$.

Spec.$(N) =$ the set of Prime ideals of $N$, Ass.$(NN) =$ The set of Associated Prime ideals of $N$ named as right $N$-Module $N$ over iteself, Min.Spec$(N) =$ Set of all minimal ideals of $N$

Let $N$ be a right Noetherian Near-Ring. For any uniform right $N$-Module $J$, the Assassinator of $J$ is denoted by Assas$(J)$. Let $M$ be a right $N$-Module.

Consider, set assas$(J) / J$ is a uniform right $N$-sub module of $M = A(MN)$. One more class of Boolean Regular $d$-Near-Rings is a Noetherian Regular $d$-Near-Rings.
**Definition 1.1.** A Proper saturated set $S$ is said to be maximal if $S$ is not contained in any proper saturated set of $N$.

i.e., there does not exist $S$ subset of $T$ is subset of $R$ implies either $S = T$ or $T = R$.

**Definition 1.2.** Let $N$ be a Commutative Ring. Let $N$ be a Noetherian Regular $\delta$-Near Ring if each $P \in A(NN)$ is strongly prime i.e., $P$ is a $\delta$-Near-Ring of $N$.

**Example 1.3.** If $N$ is a right Module over $N$ itself, we note that $Ass(N_N) = A(N_N)$ [6Y of Goodearly and warfield[ 4]]

The article concerns the study of skew polynomial rings over Noetherian Regular $\delta$-Near Rings. Let $N$ be a ring, $s$ be an endomorphism of $N$ and $d$ a $\delta$-derivation of $N$ such that $d : N \in N$ is an additive map with $\delta(ab) = \delta(a) \sigma(b) + a \delta(b)$ for every $a, b \in N$.

**Example 1.4.** Let $\sigma$-derivation be an endomorphism of a Noetherian near ring $N$ and $\delta : N \rightarrow N$ be any mapping. Let $\phi : N \rightarrow M_2(N)$ defined by $\phi(\sigma n) = [\sigma(n)0 \mid \delta(a)n]$, for every $n \in N$ be a Ring homomorphism Then, $\delta$ is a $\sigma$-derivation of $N$.

Let $\phi(N)$ is Ore Extension $N[x; \sigma, \delta]$. If $I$ be an ideal of $N$ such that $I$ is a $\sigma$ stable. i.e., $\sigma(I) = I$ and $I$ is a $\delta$-invariant ideal of a Noetherian Near Ring. i.e., $\delta(I) \subseteq I$ then $I[x; \sigma, \delta] = \phi(I)$.

Let $N[x; \sigma, \delta] = \text{set of Polynomials with co-efficients in } N$ i.e., $\{ \sum x^i a_i, a_i \in N \}$ in which '$.' is subject to realization $ax = x \sigma(a) + \delta(a)$, for all $a \in N$.

**Example 1.5.** Let $N = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where $F$ is a field. Then $P(N) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ let, $\sigma : N \rightarrow N$ be defined by, $\sigma((a b = a 0 0 c)) 0 c$

It can be seen that $a \sigma$ endomorphism of $N$ and $N$ is a $\sigma(^*)$-Ring or Noetherian Regular $\delta$-Near-Ring.

**Definition 1.6.** An Integral Domain $N$ with Quotient field $F$ is called a Noetherian Regular $\delta$-Near Ring (NR-$\delta$-NR) if each Prime ideal $P$ of $N$ is strongly Prime i.e., $(ab \in P, a \in F, b \in F \Rightarrow \text{either } a \in P \text{ or } b \in P)$
Example 1.7. Let $F = Q(\sqrt{2})$ set $V = F + x F[[x]] = F[[x]]$ Then $V$ is a Noetherian Regular $\delta$-Near-Ring. Let $S = Q + Qx + x^2$ $V$ is not a Noetherian Regular $\delta$-Near Ring.

Definition 1.8. (i) A Prime ideal $P$ of $N$ is said to be strongly prime if $aP, bP$ are Comparable ideals i.e., $aP \subseteq bP$ or $bN \subseteq aP$ for all $a, b \in N$.

(ii) A ring $N$ is said to be Noetherian Regular $\delta$-Near-Ring(NR-$\delta$-NR) if each prime ideal $P$ of $N$ is strongly prime and denoted by $S.Spec(N)$=strongly prime ideals.

Theorem 1.9. Let $N$ be a commutative Noetherian Regular Near-Ring(NRNR) which is also an Algebra over $Q$. Let $\sigma$ be an automorphism of $N$ such that $N$ is a $\sigma$ (*)-ring and $\delta$ a $\sigma$-derivation of $N$ that is for $U \in S.Spec.(N)$ With $\sigma(U) = U$ and $\delta(U) \subseteq U \rightarrow o(U) \in S.Spec.(o(N))$. Then $N$ is a Noetherian Regular $\delta$-Near-Ring(NR-$\delta$-NR). This implies $o(N)$ is a Noetherian Regular $\delta$-Near Ring(NR-$\delta$-NR).

Definition 1.10. We recall that a Prime ideal $P$ of $N$ is said to be divided if it is Comparable to every ideal of $N$. A ring $N$ is called a divided ring if every Prime ideal of $N$ is divided. (Badawi[1]) It is known as Lemma (1) of Badawi,Anderson,Dobbs[2]) that a Noetherian Regular $\delta$-Near-Ring(NR-$\delta$-NR) is a divided ring.

Theorem 1.11. Let $N$ be a commutative Noetherian Regular Near-Ring(NRNR) which is also an Algebra over $Q$. Let $\sigma$ be an automorphism of $N$ such that $N$ is a $\sigma$ (*)-ring and $\delta$ a $\sigma$-derivation of $N$. Then $N$ is almost $\delta$ -divided ring $\rightarrow$ that $o(N)$ is an almost $\delta$-divided ring.

2 Main Results

One more class of Prime Ideals of Noetherian Regular $\delta$-Near Ring (NR-$\delta$-NR) is the set of Associated Prime Ideals and we are therefore motivated to investigate the above results for Associated Prime Ideals or the Assassinators of a Noetherian Regular $\delta$-Near-Rings(NR-$\delta$-NR).

Example 2.1. A semi prime Noetherian Near Ring is a Semi Noetherian Regular $\delta$-Near-Ring.(SNR-$\delta$-NR).

Definition 2.2. Let $N$ be a ring we say that $N$ is a semi $\sigma$ -divided ring (semi - $\delta$ -divided) ring if each $P \in A(N_N)$.

Theorem 2.3. Let $N$ be a semi Prime commutative Noetherian $Q$-Algebra $\sigma$ be an automorphism of $N$ such that $N$ is a $\sigma$ (*)-ring and $\delta$ a $\sigma$-derivation
Noetherian regular $\delta$-near rings

of $N$ such that $\sigma \delta (a) = \delta (\sigma (a))$ for every $a \in N$. Then (i) if for all $U \in S.\text{Spec}(N)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U \rightarrow S.\text{Spec}(o(N))$ Then $N$ is a semi Noetherian Regular $\delta$-Near-Ring (SNR-$\delta$-NR) $\rightarrow N[x; \sigma, \delta]$ is Semi Noetherian Regular $\delta$-Near-Ring (SNR-$\delta$-NR), (ii) If $N$ is Semi-derided ring, Then $o(N)$ is a Semi Noetherian Regular $\delta$-Near-Ring (SNR-$\delta$-NR)

Proof : See Prop.(2.1) of Bhat [6]

Theorem 2.4. Let $N$ be a Noetherian Ring which is also an Algebra over $Q$. Let $\sigma$ be an automorphism of $N$ such that $N$ is a $\sigma(*)$-ring and $\delta$ a $\sigma$-derivation of $N$. Then (i) if $U$ is a minimal Prime ideal of $N$, Then $o(U)$ is a minimal prime ideal of $o(N)$ and $o(U) \cap N = U$ and (ii) if $P$ is a minimal prime ideal of $o(N)$ then $P \cap N$ is a minimal prime ideal of $N$.

Proof : See lemma (2.2) of Bhat [6]

Note : The lemma is true even if $N$ is non-commutative.

Theorem 2.5. Let $N$ be a right or left Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR) .Let $s$ be an automorphism of $N$ and $d$ as $\delta-$ derivation of $N$.Then the Ore Extension $o(N) = N[x; s, d]$ is also right or left Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR).

Proof : See theorem [2.6] of Goodearl and Warfield [4] It is known theorem (2.6) of Bhat [6] that if $N$ is a commutative Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR)where $x \notin P$, for all $P \in S.\text{Spec}(S(N))$. Then $S(N)$ is also Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR).

It is also known (Theorem (2.10) of Bhat [6]) that if $N$ is a commutative Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR) of Q-Algebra which is also a Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR).

Then $D(N)$ is also a Divided Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR). These results have been generalized for $o(N)$ over Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR) and in V K Bhat [6].

Let $N$ be a Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR). We know that $\text{Ass}(N_N)$ is finite, $\sigma^j(U) \in \text{Ass}(N_N) \forall U \in \text{Ass}(N_N) \forall j \geq 1, \exists$ an integer $m$ suchthat $\sigma^m(U) = U, \forall U \in \text{Ass}(N_N)$.

We denote, $\cap \sigma^j(U) = U(0), j = 1 \text{ to } m$ and Since, min. $\text{Spec}(N) \leq \text{ in finite}$ (finite) and also same notation for min. $\text{spec}(N)$.

Hence proved the theorem.

Theorem 2.6. Let $N$ be a Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR) and $\sigma$ be an automorphism of $N$. Then (i) $P \in \text{Ass}(S(N)S(N))$ $\iff \exists Q \in \text{Ass}(N_N)$ such that $s(P \cap N) = P$ and $P \cap N = Q_0$ (ii) $P \in \text{Min.}\text{Spec}(S(N))$ $\iff \exists Q \in \text{Min.}\text{Spec}(N)$ such that $S(P \cap N) = P$ and $P \cap N = Q_0$.

Proof : see theorem (2.4) of Bhat [6] Now we can give an analogue of theorem (1.3) for semi Noetherian Regular $\delta$-Near Ring (SNR-$\delta$-NR).
Theorem 2.7. Let $N$ be a Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR) which is also an algebra over $Q$. Let $\sigma$ be an automorphism of $N$ is a as $\sigma$ (*)-ring and $\delta$ a $\sigma$-derivation of $N$. further , Let any $U \in S\text{Spec}(N)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U \rightarrow o(U) \in S\text{Spec}(o(N))$. Then $o(N)$ is a Noetherian Regular $\delta$-Near Ring (NR-$\delta$-NR).

Proof : $o(N)$ is a Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR) by theorem (2.5). We note that $x \notin P$ for any Prime ideal $P$ of $o(N)$, Ore Extension of $N$, as it is not a zero divisor. Let,$J \in A(o(N))$. Say, $J = \text{Ann}(I) = \text{Assas}(I)$ for some ideal $I$ of $o(N)$ such that $I$ is uniform as a right $o(N)$-Module. Then, $J \in \text{MinSpec}(o(N))$ where $\text{MinSpec}(o(N))= \{ \text{set of Minimal prime ideals generated by } o(N) \text{ of } N \}$. By Remark (2.1), we have $A(o(N)o(N)) = \text{Assas}(o(N)o(N))$.

By lemma (2.4), $J \cap N \in \text{MinSpec}(N)$. Also, $\sigma(J \cap N) = J \cap N$ and $\delta(J \cap N) \in J \cap N$. Now, $J \in A(o(N)o(N)) \in J \cap N$ and therefore , using the fact that $\sigma(J \cap N) = J \cap N$ and $\delta(J \cap N) \subseteq J \cap N$. And by known theorem(2.6)we get that $J \cap N = \text{Assas}(N_N) = A(N_N)$.

As we note that $(J \cap N)_{o0} = J \cap N$. Now, N is a Semi-Noetherian Regular delta-Near-Ring(SNR-$\delta$-NR) $Q$-Algebra. Therefore, $J \cap N \in S\text{Spec}(N)$.

By hypothesis, $o(J \cap N) \in S\text{Spec}(o(N))$ where, $S\text{Spec}(o(N))$ = set of semi prime ideals of Ore Extension of $N$ And Ore extension of $N$ = set skew polynomial rings, Differential operator rings.

Further, it is easy to see that, $o(J \cap N) = J$ since, $(J \cap N) \subseteq J$ or $N$.

Therefore, $J \in S\text{Spec}(o(N))$ where, $S\text{Spec}(o(N))$ = set of semi prime ideals of Ore Extension of $N$.Therefore, $o(N)$ is a Noetherian Regular $\delta$-Near Ring (NR-$\delta$-NR).

For all $U \in S\text{Spec}(N)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U \rightarrow o(U) \in S\text{Spec}(o(N))$ cannot be deleted as extension of a strongly Prime ideal of $N$ need not be a strongly Prime Ideals of $o(N)$.

Hence Proved the Theorem.

Example : (By Example 4 of Bhat [6]) Let us suppose, $N = Z(P)$ . This infact discrete Noetherian Regular $\delta$-Near Ring (NR-$\delta$-NR) and therefore Ideal $P = PN$ is strongly Prime. But, $PN[x]$ is not strongly Prime in $N[x]$ be can it is not Comparable with $xN[x]$. Therefore, condition of being strongly Prime in $N[x]$ fails for $a = 1$, $b = x$.

Lemma : Let $N$ be a semi prime commutative Noetherian Regular-Near-Ring(SNR-$\delta$-NR) which is also a algebra over $Q$. Let , $\sigma(U) = U$ and $\forall U \in A(N_N))$. Further, Let any $U \in S\text{Spec}(N)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U \rightarrow o(U) \in S\text{Spec}(o(N))$. Then $o(N)$ is a Noetherian Regular $\delta$-Near-Ring (NR-$\delta$-NR). $A(N_N)$. Further, Let any $U \in S\text{Spec}(N)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U \rightarrow o(U) \in S\text{Spec}(o(N))$.

Proof : Given $N$ be a semi Prime commutative Noetherian Regular-Near-Ring (SNR-$\delta$-NR) Algebra over $Q$. Let $o(N)$ is a Noetherian Regular-Near-
Let $J \in (o(N)o(N))$. Then as in theorem (2.7) above $J \cap N \subseteq A(N_N) = \text{Ass}(N_N)$.

Now, $\sigma(J \cap N) = J \cap N$.

Therefore, by Lemma (2.6) of Bhat [6] in $\sigma$-derivation of $N$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in N$ and $N$ Then $N$ is a Semi Noetherian Regular $\delta$-Near-Ring $\rightarrow o(N)$ is a Noetherian Regular Semi $\delta$-Near-Ring $\rightarrow o(N)$ algebra.

These results have been generalised for $o(N)$ over almost $\delta$-divided rings in Bhat [6] as mentioned in theorem (1.4). Hence proved the Theorem.

Theorem 2.8. Let $N$ be a Semi Prime Commutative Noetherian Regular $\delta$-Near-Ring $\rightarrow o(N)$ algebra over $Q$. Let $\sigma$ be an automorphism of $N$ such that $N$ is a $\sigma(*)$-ring and $\delta$ is a $\sigma$-derivation of $N$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in N$ and $N$ Then $N$ is a Semi Noetherian $\delta$-Near-Ring $\rightarrow o(N)$ is a Noetherian Regular Semi $\delta$-Near-Ring $\rightarrow o(N)$.

Proof: Let $o(N)$ be a Noetherian Regular $\delta$-Near-Ring $\rightarrow o(N)$ algebra by known Theorem. Let $\sigma$ can be extended an automorphism of $o(N)$ such that $\sigma(x) = x$ and $\delta$ can be extended to a $\sigma$-derivation of $o(N)$ such that $\delta(x) = 0$.

Let $J \in A(o(N)\sigma(N))$ and $0 \in K$ be a proper ideal of $o(N)$ such that $\sigma(K) = K$ and $\delta(K) \subseteq K$. Now by theorem (2.7), we have $\sigma(J \cap N) = J \cap N \delta (J \cap N) \subseteq J \cap N$. So, $J \cap N \subseteq \text{Ass}(N_N) = A(N_N)$. Also, $K \cap \text{NisanidealofNwith}\sigma(K \cap N) = K \cap N \delta (K \cap N) \subseteq K \cap N$. Now, $N$ is a Semi $\delta$-divided ring and therefore, $J \cap N \subseteq K \cap N$. Therefore, $o(J \cap N) \subseteq o((K \cap N)) \subseteq J \subseteq K$.

Therefore, $o(N)$ is a Noetherian Regular Semi $\delta$-Near-Ring $\rightarrow o(N)$ algebra. Hence proved the Theorem.

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