Qualitative Behavior of a Fourth-Order Rational Difference Equation

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Abstract

In this paper, we investigate the qualitative behavior of the fourth-order difference equation

\[ x_{n+1} = \frac{(x_{n-1}x_{n-2} + x_{n-3})x_{n-3} + 1)}{(x_{n-2}x_{n-3} + 1)} \]

for \( n = 0, 1, \ldots \)

where \( a \in (0, \infty) \) and the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty) \).

Keywords: Difference equation, Stability, Semi-cycle, Globally stable

1 Introduction

Recently there has been a great interest in studying the qualitative behaviors of rational difference equations. Berenhaut, Foley and Stevic [1] has showed that the unique positive equilibrium \( \bar{y} = 1 \) of the difference equation \( y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}} \) is globally asymptotically stable. Y. Chen and X. Li [2] investigated the dynamical properties of the fourth-order nonlinear difference equation \( x_{n+1} = \frac{(x_{n-1}^a + x_{n-3})x_{n-3} + 1)}{(x_{n-2}^a x_{n-3} + 1)} \) with
nonnegative initial conditions and \( a \in [0,1) \). For more related work see [4-7].

To be motivated by the above studies, in this paper, we consider the following fourth-order nonlinear difference equation:

\[
x_{n+1} = \frac{x_{n-1}^a x_{n-2} + x_{n-1} x_{n-3}^a + 1}{x_{n-2}^a + x_{n-3}^a + 1}, \quad n = 0,1,\ldots
\]

(1.1)

where \( a \in (0,\infty) \) and the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty) \). It is easy to see that the positive equilibrium \( \bar{x} = 1 \) of Eq.(2.1) satisfies \( \bar{x} = (2\bar{x}^{-1} + 1)/(2\bar{x} + 1) \).

**Definition 1.1:** A positive semi-cycle of a solution \( \{x_n\}_{n=-3}^\infty \) of Eq. (1.1) consists of a “string” of terms \( \{x_i, x_{i+1}, \ldots, x_m\} \) all greater than or equal to the equilibrium \( \bar{x} \), with \( \ell \geq -3 \) and \( m \leq \infty \) such that (either \( \ell = -3 \) or \( \ell > 3 \) and \( x_{\ell+1} < \bar{x} \)) and (either \( m = \infty \) or \( m < \infty \) and \( x_{m+1} < \bar{x} \)). A negative semi-cycle of a solution \( \{x_n\}_{n=-3}^\infty \) of Eq. (1.1) consists of a “string” of terms \( \{x_i, x_{i+1}, \ldots, x_m\} \) all less than \( \bar{x} \), with \( \ell \geq -3 \) and \( m \leq \infty \) such that (either \( \ell = -3 \) or \( \ell > 3 \) and \( x_{\ell+1} \geq \bar{x} \)) and (either \( m = \infty \) or \( m < \infty \) and \( x_{m+1} \geq \bar{x} \)). The length of a semi-cycle is the number of the total terms contained in it.

**Definition 1.2:** A solution \( \{x_n\}_{n=-3}^\infty \) of Eq. (1.1) is said to be eventually trivial if \( x_n \) is eventually equal to \( \bar{x} = 1 \); Otherwise is said to be nontrivial. A solution \( \{x_n\}_{n=-3}^\infty \) of Eq. (1.1) is said to be eventually positive (negative) if \( x_n \) is eventually greater (less) than \( \bar{x} = 1 \).

**2 Three Lemmas**

**Lemma 2.1.** A positive solution \( \{x_n\}_{n=-3}^\infty \) of Eq. (1.1) is eventually equal to 1 if and only if

\[
(x_{-1} - 1)(x_{-2} - 1) = 0
\]

(2.1)

**Proof.** Assume that (2.1) holds.

(a) if \( x_{-1} = 1 \), then \( x_n = 1 \) for \( n \geq 1 \)

(b) if \( x_{-2} = 1 \), then \( x_n = 1 \) for \( n \geq 14 \).

Conversely, assume that

\[
(x_{-1} - 1)(x_{-2} - 1) \neq 0
\]

(2.2)

Then one can show that \( x_n \neq 1 \) for any \( n \geq 1 \). Assume that some \( N \geq 1 \), \( x_N = 1 \) and that \( x_n \neq 1 \) for \( -2 \leq n \leq N - 1 \).

(2.3)

It is easy to see that

\[
1 = x_N = (x_{N-2} x_{N-3} + x_{N-2} x_{N-4})/(x_{N-3} + x_{N-4} + 1)
\]

which implies

\[
(x_{N-3} + x_{N-4})(1 - x_{N-2}) = 0.
\]

Obviously, this contradicts (2.3).

**Remark 2.1.** If the initial conditions do not satisfy Eq. (1.1), then, for any
solution \( \{x_n\} \) of Eq. (1.1), \( x_n \neq 1 \) for \( n \geq -3 \). Here, the solution is a nontrivial one.

**Lemma 2.2.** Let \( \{x_n\}_{n=-3}^{\infty} \) be a nontrivial positive solution of Eq. (1.1). Then the following conclusions are true for \( n \geq 0 \):

(a) \( (x_{n+1}-1)(x_{n-1}-1) > 0 \)

(b) \( (x_{n+1} - x_{n-1})(x_{n-1} - 1) < 0 \)

(c) \( (x_{n+1} - 1)(x_{n-1} - 1)(x_{n-3} - 1) > 0 \)

**Lemma 2.3.** If \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (1, \infty) \), then \( \{x_n\}_{n=-3}^{\infty} \) has a positive semi-cycle with an infinite number of terms and it monotonically tends to the positive equilibrium point \( x_1 \).

**Proof.** If \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (1, \infty) \), from Lemma 2.2.(a) and (b), for \( n \geq -3 \)

\[
1 < x_{2k+1} < ... < x_1 < x_{-1} \quad \text{and} \quad 1 < x_{2k} < ... < x_2 < x_0
\]

Clearly, \( \{x_n\}_{n=-3}^{\infty} \) has a positive semicycle with an infinite number of terms and monotonically decreasing for \( n \geq 0 \). So the limit \( \lim_{n \to \infty} x_n = L \) exists and finite.

Taking the limits on both sides of Eq. (1.1), we have \( L = \left(2L^{x_{1L}} + 1\right) / \left(2L^{1} + 1\right) \).

**3 Main Results and their Proofs**

Here we confine us to consider the situation of the strictly oscillatory solution of Eq. (1.1).

**Theorem 3.1.** Let \( \{x_n\}_{n=-3}^{\infty} \) be a strictly oscillatory solution of Eq. (1.1). Then the rule for the lengths of positive and negative semi-cycles of this solution to successively occur is \( ...,1^+, 1^-, 3^+, 1^-, 1^+, 3^-, 1^-, ..., \) or \( ..., 2^-, 1^+, 2^-, 1^+, 2^-, 1^+, ..., \) or \( ..., 2^+, 4^-, 2^+, 4^-, 2^+, 4^-, ..., \).

**Proof.** By Lemma 2.3. (c), one can see the length of a positive semi-cycle is not larger than 3 and the length of a negative semi-cycle is at most 4. Based on the strictly oscillatory character of the solution, for some \( p \geq 0 \), that one of the following four cases must occur:

- **Case 1.** \( x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1 \) and \( x_p > 1 \)
- **Case 2.** \( x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1 \) and \( x_p > 1 \)
- **Case 3.** \( x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1 \) and \( x_p < 1 \)
- **Case 4.** \( x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1 \) and \( x_p < 1 \)

If Case 1 occurs, it follows from Lemma 2.2.(c) that
It means that the rule of the lengths of positive and negative semi-cycles of the solution of Eq. (1.1) to occur successively is ...,1,1,3,1,1,1,3,1,....

If Case 2 occurs, it follows from Lemma 2.2.(c) that

It means that the rule of the lengths of positive and negative semi-cycles of the solution of Eq. (1.1) to occur successively is ...,2,1,2,1,2,1,....

If Case 3 occurs, it follows from Lemma 2.2.(c) that

This shows that the rule for the numbers of terms of positive and negative semi-cycles of the solution of Eq. (1.1) to successively occur is ...,2+,4−,2+,4−,2+,4−,...

When Case 4 happens, a similar deduction leads to that the regulation for the lengths of positive and negative semi-cycles of the solution of Eq. (1.1) to occur successively is ...,1+,3−,1−,1+,3−,1−,....

**Theorem 3.2.** Assume that \( a \in (0, \infty) \). Then the positive equilibrium of Eq. (1.1) is globally asymptotically stable.

**Proof.** We must prove that the positive equilibrium point \( \bar{x} \) of Eq. (1.1) is both locally asymptotically stable and globally attractive. The linearized equation of Eq. (1.1) is

\[
y_{n+1} = 0.y_n + \frac{2}{3}y_{n-1} + 0.y_{n-2} + 0.y_{n-3} , \quad n = 0,1,\ldots
\]

From [3, Remark 1.3.7], \( \bar{x} \) is locally asymptotically stable. It remains to verify that every positive solution \( \{x_n\}_{n=3}^{\infty} \) of Eq. (1.1) converges to 1 as \( n \to \infty \).

Namely, we want to prove

\[
\lim_{n \to \infty} x_n = \bar{x} = 1
\]  

(3.1)

If the solution is nonoscillatory about the positive equilibrium point \( \bar{x} \) of Eq. (1.1), then from Lemma 2.1 and Lemma 2.3, the solution is either equal to 1 or eventually positive one which has an infinite number of terms and monotonically tends to the positive equilibrium point \( \bar{x} \) of Eq. (1.1), and so Eq (3.1) holds.
Consider now \( \{x_n\} \) to be strictly oscillatory about the positive equilibrium point \( x^* \) of Eq. (1.1).

By virtue of Theorem 3.1, one understands that the rule for the lengths of positive and negative semi-cycles which occur successively is \( 1^-, 3^+, 1^-, 1^-, 3^+, 1^-, \ldots \) or \( 2^-, 1^-, 2^-, 1^+, 2^-, \ldots \) or \( 2^+, 4^-, 2^+, 4^-, 2^+, \ldots \) or \( 2^-, 4^-, 2^-, 4^-, 2^+, \ldots \) or \( 1^-, 3^+, 1^-, 1^-, 3^+, 1^-, \ldots \) or \( 1^-, 3^+, 1^-, 1^-, 3^+, 1^-, \ldots \).

First, we investigate the case is \( 1^-, 3^+, 1^-, 1^-, 3^+, 1^-, \ldots \).

For simplicity, we denote by \( \{x_p\}^+ \) the terms of a positive semi-cycle of length one, followed by \( \{x_{p+1}\}^- \) a negative semi-cycle with length one, then a positive semi-cycle \( \{x_{p+2}, x_{p+3}, x_{p+4}\}^+ \) and a negative semi-cycle \( \{x_{p+5}\}^- \), and so on.

Namely, the rule for the lengths of positive and negative semi-cycles to occur successively can be periodically expressed as follows:
\[
\{x_{pn+6}\}^+, \; \{x_{pn+6n}^-\}, \; \{x_{pn+6n+1}, x_{pn+6n+3}, x_{pn+6n+4}\}^+, \; \{x_{pn+6n+5}\}^-,
\]
\( n=0, 1, \ldots \)

then the following results can be easily observed:

(i) \( x_{pn+6n+6} < x_{pn+6n+4} < x_{pn+6n+2} < x_{pn+6n} \);
(ii) \( x_{pn+6n+3} > 1, \; x_{pn+6n+1} < 1 \).

Inequality (i) can be easily seen from Lemma 2.2(b) for \( n=0, 1, \ldots \).

From the observations of
\[
x_{pn+6n+6} = \frac{x_{pn+6n+3}a^x_{pn+6n+2} + x_{pn+6n+3}^a x_{pn+6n+1} + 1}{x_{pn+6n+3} x_{pn+6n+2} + x_{pn+6n+1} + 1} \geq \frac{x_{pn+6n+3} x_{pn+6n+2} + x_{pn+6n+3} x_{pn+6n+1} + 1}{x_{pn+6n+3} (x_{pn+6n+2} + x_{pn+6n+1}) + 1}
\]

Similarly \( x_{pn+6n+3} x_{pn+6n+1} < 1 \) can be shown. Combining the above inequalities, one can derive
\[
x_{pn+6n+1} < \frac{1}{x_{pn+6n+3}} < x_{pn+6n+5} < 1 \quad (3.2)
\]
\[
1 < x_{pn+6n+6} < x_{pn+6n+4} < x_{pn+6n+2} < x_{pn+6n} \quad (3.3)
\]

It follows from (3.2) that \( \{x_{pn+6n+1}\} \) is increasing with upper bound 1. So, the limit
\[
limit_{n \to \infty} x_{pn+6n+1} = L \quad (3.4)
\]
exists and finite. Accordingly, by view of (3.2), we obtain
\[
lim x_{pn+6n+5} = L \quad \text{and} \quad \lim x_{pn+6n+3} = \frac{1}{L} \quad (3.5)
\]

It is easy to see from (3.3) that \( \{x_{pn+6n}\} \) is decreasing with lower bound 1. So, the limit
\[
\lim_{n \to \infty} x_{p+6n} = M
\]
exists and finite. Accordingly, by view of (3.3), we obtain
\[
\lim_{n \to \infty} x_{p+6n+2} = \lim_{n \to \infty} x_{p+6n+4} = \lim_{n \to \infty} x_{p+6n+6} = M
\]
Taking the limits on both sides of
\[
x_{p+6n+6} = \frac{x_{p+6n+4}x_{p+6n+3} + x_{p+6n+4}x_{p+6n+2} + 1}{x_{p+6n+3} + x_{p+6n+2} + 1},
\]
has \( M = \left( ML^2 + M^2 + 1\right) / \left( L^2 + M^2 + 1\right) \), which gives rise to \( M = 1 \).
Taking the limits on both sides of
\[
x_{p+6n+4} = \frac{x_{p+6n+3}x_{p+6n+2} + x_{p+6n+3}x_{p+6n+1} + 1}{x_{p+6n+2} + x_{p+6n+1} + 1},
\]
has \( L = \left( \frac{1}{L} M^2 + \frac{1}{L} L^2 + 1\right) / \left( M^2 + L^2 + 1\right) \), which gives rise to \( L = 1 \).
So we can easily see that
\[
\lim_{n \to \infty} x_{p+6n+k} = 1, \quad k = 0,1,2,3,4,5,6
\]
For ..., \( 2^-, 1^+, 2^-, 1^+, 2^-, ..., \) and ..., \( 2^-, 4^-, 2^+, 4^-, 2^+, 4^-, ..., \) can be similarly shown.

References


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