Common Fixed Point Theorems

in Fuzzy Metric Spaces

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Abstract

This paper presents some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces.

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1 Introduction

Zadeh [23] introduced the concept of Fuzzy set in 1965 and in the next decade Kramosil and Michalek [12] introduced the concept of fuzzy metric space in 1975. George and Veermani [5] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki [22] proved fixed point theorems for R-weakly commutating mappings. Pant [16, 17, 18] introduced the new concept reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam et al.[3], have shown that Rhoades [20] open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [18] obtained some analogous results proved by Balasubramaniam et al. Recent literature on fixed point in fuzzy metric space can be viewed in [1, 7, 14]. This paper presents some common fixed point theorems for more general commutative condition i.e. occasionally weakly compatible mappings in fuzzy metric space.
2 Preliminary Notes

Definition 2.1 [23] A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0, 1]$.

Definition 2.2 [21] A binary operation $\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norms if $\ast$ is satisfying conditions:
(i) $\ast$ is an commutative and associative;
(ii) $\ast$ is continuous;
(iii) $a \ast 1 = a$ for all $a \in [0, 1]$;
(iv) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Definition 2.3 [5] A 3-tuple $(X, M, \ast)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $\ast$ is a continuous t-norm and $M$ is a fuzzy set of $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$,
(f1) $M(x, y, t) > 0$;
(f2) $M(x, y, t) = 1$ if and only if $x = y$
(f3) $M(x, y, t) = M(y, x, t)$;
(f4) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$;
(f5) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.
Then $M$ is called a fuzzy metric on $X$. Then $M(x, y, t)$ denotes the degree of nearness between $x$ and $y$ with respect to $t$.

Example 2.4 Let $(X, d)$ be a metric space. Define $a \ast b = ab$ { or $a \ast b = \min(a, b)$ } for all $x, y \in X$ and $t > 0$,
$$M(x, y, t) = \frac{t}{t + d(x, y)}$$
Then $(X, M, \ast)$ is a fuzzy metric space and the fuzzy metric $M$ induced by the metric $d$ is often referred to as the standard fuzzy metric.

Definition 2.5 [5]: Let $(X, M, \ast)$ be a fuzzy metric space. Then
(a) a sequence $\{x_n\}$ in $X$ is said to converges to $x$ in $X$ if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.
(b) a sequence $\{x_n\}$ in $X$ is said to be Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_m, x_n, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.
(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6 [22] A pair of self-mappings $(f, g)$ of a fuzzy metric space $(X, M, \ast)$ is said to be
(i) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for all $x \in X$ and $t > 0$.
(ii) R-weakly commuting if there exists some $R > 0$ such that $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ for all $x \in X$ and $t > 0$. 
Definition 2.7 [8] Two self mappings \( f \) and \( g \) of a fuzzy metric space \((X, M, \ast)\) are called compatible if\\( \lim_{n \to \infty} M(fg x_n, gfx_n, t) = 1 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that\\( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x \) for some \( x \) in \( X \).

Definition 2.8 [3]: Two self maps \( f \) and \( g \) of a fuzzy metric space \((X, M, \ast)\) are called reciprocally continuous on \( X \) if\\( \lim_{n \to \infty} fg x_n = fx \) and\\( \lim_{n \to \infty} gfx_n = gx \) whenever \( \{x_n\} \) is a sequence in \( X \) such that\\( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x \) for some \( x \) in \( X \).

Lemma 2.9 Let \((X, M, \ast)\) be a fuzzy metric space. If there exists \( q \in (0, 1) \) such that\\( M(x, y, qt) \geq M(x, y, t) \) for all \( x, y \in X \) and \( t > 0 \), then \( x = y \).

Definition 2.10 Let \( X \) be a set, \( f, g \) self maps of \( X \). A point \( x \) in \( X \) is called a coincidence point of \( f \) and \( g \) iff \( fx = gx \). We shall call \( w = fx = gx \) a point of coincidence of \( f \) and \( g \).

Definition 2.11 [9] A pair of maps \( S \) and \( T \) is called weakly compatible pair if they commute at coincidence points.

The concept occasionally weakly compatible is introduced by M. Al-Thagafi and Naseer Shahzad [2]. It is stated as follows.

Definition 2.12 Two self maps \( f \) and \( g \) of a set \( X \) are occasionally weakly compatible (owc) iff there is a point \( x \) in \( X \) which is a coincidence point of \( f \) and \( g \) at which \( f \) and \( g \) commute.

A. Al-Thagafi and Naseer Shahzad [2] shows that occasionally weakly is weakly compatible but converse is not true.

Example 2.13 [2] Let \( R \) be the usual metric space. Define \( S, T : R \to R \) by \( Sx = 2x \) and \( Tx = x^2 \) for all \( x \in R \). Then \( Sx = Tx \) for \( x = 0, 2 \) but \( ST0 = TS0 \), and \( ST2 \neq TS2 \). \( S \) and \( T \) are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.14 [10] Let \( X \) be a set, \( f, g \) owc self maps of \( X \). If \( f \) and \( g \) have a unique point of coincidence, \( w = fx = gx \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

3 Main Results

Theorem 3.1 Let \((X, M, \ast)\) be a complete fuzzy metric space and let \( A, B, S \) and \( T \) be self-mappings of \( X \). Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc. If there exists \( q \in (0, 1) \) such that
\[ M(Ax, By, qt) \geq \phi \left[ \min \{M(Sx, Ty, t), M(Sx, Ax, t)\} \ast \min \{M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \right] \quad \ldots \ldots \ (1) \]

for all \( x, y \in X \) and \( \phi : [0, 1] \to [0, 1] \) such that \( \phi(t) > t \) for all \( 0 < t < 1 \), then there exists a unique common fixed point of \( A, B, S \) and \( T \).

**Proof:** Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc, so there are points \( x, y \in X \) such that \( Ax = Sx \) and \( By = Ty \). We claim that \( Ax = By \). If not, by inequality (1)

\[ M(Ax, By, qt) \geq \phi \left[ \min \{M(Sx, Ty, t), M(Sx, Ax, t)\} \ast \min \{M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \right] \]

\[ = \phi \left[ \min \{M(Ax, By, t), M(Ax, Ax, t)\} \ast M(By, By, t) \right] \]

\[ = \phi \left[ M(Ax, By, t) \right] > M(Ax, By, t) \]

Therefore \( Ax = By \), i.e. \( Ax = Sx = By = Ty \). Suppose that there is another point \( z \) such that \( Az = Sz \) then by (1) we have \( Az = Sz = By = Ty \), so \( Ax = Az \) and \( w = Ax = Sx \) is the unique point of coincidence of \( A \) and \( S \). By Lemma 2.14 \( w \) is the only common fixed point of \( A \) and \( S \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \).

Assume that \( w \neq z \). We have

\[ M(w, z, qt) = M(Aw, Bz, qt) \]

\[ \geq \phi \left[ \min \{M(Sw, Tz, t), M(Sw, Az, t)\} \ast \min \{M(Bz, Tz, t), M(Aw, Tz, t), M(Bz, Sw, t)\} \right] \]

\[ = \phi \left[ \min \{M(w, z, t), M(w, z, t)\} \ast M(z, z, t) \right] \]

\[ = \phi \left[ M(w, z, t) \right] > M(w, z, t) \]

Therefore we have \( z = w \) by Lemma 2.14 and \( z \) is a common fixed point of \( A, B, S \) and \( T \). The uniqueness of the fixed point holds from (1).

**Theorem 3.2** Let \((X, M, *)\) be a complete fuzzy metric space and let \( A, B, S \) and \( T \) be self-mappings of \( X \). Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc. If there exists \( q \in (0, 1) \) such that

\[ M(Ax, By, qt) \geq \left[ \min \{M(Sx, Ty, t), M(Sx, Ax, t)\}, M(By, Ty, t) \right] \ast \min \{M(Ax, Ty, t), M(By, Sx, t)\} \quad \ldots \ldots \ (2) \]
for all \( x, y \in X \) and for all \( t > 0 \), then there exists a unique point \( w \in X \) such that \( Aw = Sw = w \) and a unique point \( z \in X \) such that \( Bz = Tz = z \). Moreover, \( z = w \), so that there is a unique common fixed point of \( A, B, S \) and \( T \).

**Proof:** Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc, so there are points \( x, y \in X \) such that \( Ax = Sx \) and \( By = Ty \). We claim that \( Ax = By \). If not, by inequality (2)

\[
M(Ax, By, qt) \geq \left[ \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\} \right. \nonumber \\
\left. \min\{M(Ax, Ty, t), M(By, Sx, t)\} \right] \\
= \left[ \min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t)\} \right. \\
\left. \min\{M(Ax, By, t), M(By, Ax, t)\} \right] \\
= M(Ax, By, t),
\]

Therefore \( Ax = By \), i.e. \( Ax = Sx = By = Ty \). Suppose that there is another point \( z \) such that \( Az = Sz \) then by (2) we have \( Az = Sz = By = Ty \), so \( Ax = Az \) and \( w = Ax = Sx \) is the unique point of coincidence of \( A \) and \( S \). By Lemma 2.14 \( w \) is the only common fixed point of \( A \) and \( S \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \).

Assume that \( w \neq z \). We have

\[
M(w, z, qt) = M(Aw, Bz, qt) \geq \left[ \min\{M(Sx, Tz, t), M(Sx, Az, t), M(Bz, Tz, t)\} \right. \nonumber \\
\left. \min\{M(Ax, Tz, t), M(Bz, Sw, t)\} \right] \\
= \left[ \min\{M(w, z, t), M(w, z, t), M(z, z, t)\} \right. \\
\left. \min\{M(w, z, t), M(z, w, t)\} \right] \\
= M(w, z, t)
\]

**Theorem 3.3** Let \((X, M, \ast)\) be a complete fuzzy metric space and let \( A, B, S \) and \( T \) be self-mappings of \( X \). Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc. If there exists \( q \in (0, 1) \) such that

\[
M(Ax, By, qt) \geq \phi \left\{M(Sx, Ty, t), M(Sx, By, t), M(By, Ty, t), M(Ax, Ty, t)\right\} \quad ............ (3)
\]

for all \( x, y \in X \) and \( \phi : [0, 1]^4 \rightarrow [0, 1] \) such that \( \phi(t, t, 1, t) > t \) for all \( 0 < t < 1 \), then there exists a unique common fixed point of \( A, B, S \) and \( T \).

**Proof:** Let the pairs \( \{A, S\} \) and \( \{B, T\} \) are owc, there are points \( x, y \in X \) such that \( Ax = Sx \) and \( By = Ty \). We claim that \( Ax = By \). By inequality (3) we have

\[
M(Ax, By, qt) \geq \phi \left\{M(Sx, Ty, t), M(Sx, By, t), M(By, Ty, t), M(Ax, Ty, t)\right\} \\
= \phi \left\{M(Ax, By, t), M(Ax, By, t), M(By, By, t), M(Ax, By, t)\right\}
\]
\[ = \phi \{M(Ax, By, t), M(Ax, By, t), 1, M(Ax, By, t)\} \]
\[ > M(Ax, By, t). \]

a contradiction, therefore \( Ax = By \), i.e. \( Ax = Sx = By = Ty \). Suppose that there is * another point \( z \) such that \( Az = Sz \) then by (3) we have \( Az = Sz = By = Ty \), so \( Ax = Az \) and \( w = Ax = Tx \) is the unique point of coincidence of \( A \) and \( T \). By Lemma 2.14 \( w \) is a unique common fixed point of \( A \) and \( S \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \). Thus \( z \) is a common fixed point of \( A, B, S \) and \( T \). The uniqueness of the fixed point holds from (3).

\[ M(w,z,qt) = M(Aw,Bz,qt) \geq \phi \{M(Sw,Tz,t), M(Sw,Bz,t), M(Bz,Tz,t), M(Aw,Tz,t),\} \]
\[ = \phi \{M(w,z,t), M(w,z,t), M(z,z,t), M(w,z,t),\} \]
\[ > M(w,z,t), \]

References


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