Existence and Nonexistence of Positive Weak Solutions for a Class of \((p, q)\)-Laplacian with Different Weights

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Abstract

We establish the results concerning existence and nonexistence of positive weak solutions for the nonlinear elliptic system

\[
\begin{align*}
-\Delta_{P,p} u &= \lambda a(x) v^\beta & \text{in } \Omega, \\
-\Delta_{Q,q} v &= \lambda b(x) u^\alpha & \text{in } \Omega, \\
u &= v = 0 & \text{on } \partial \Omega.
\end{align*}
\]

where \(\Delta_{R,r}\) with \(r > 1\) and \(R = R(x)\) is a weight function, denotes the weighted \(r\)-Laplacian defined by \(\Delta_{R,r}u \equiv \text{div}[R(x)|\nabla u|^{r-2}\nabla u]\), \(\lambda\) is a positive parameter, \(a(x), b(x)\) are weight functions, \(0 < \alpha < p - 1, 0 < \beta < q - 1\) and \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary \(\partial \Omega\). We use the method of sub-supersolutions to establish our results.

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1 Introduction:

In this paper, we are concerned with the existence and nonexistence of positive weak solutions for the nonlinear elliptic system

\[
\begin{align*}
-\Delta_{p,p} u &= \lambda a(x)v^\beta \quad \text{in } \Omega, \\
-\Delta_{q,q} v &= \lambda b(x)u^\alpha \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.1)

where $\Delta_{R,r}$ with $r > 1$ and $R = R(x)$ is a weight function, $R(x) = P(x)$ when $r = p$ and $R(x) = Q(x)$ when $r = q$, denotes the weighted $r$-Laplacian defined by $\Delta_{R,r} u \equiv \text{div}[R(x)|\nabla u|^{r-2}\nabla u]$, $\lambda$ is a positive parameter, $a(x)$ and $b(x)$ are weight functions and that there exist positive constants $a_0, b_0$ such that $a(x) \geq a_0$, $b(x) \geq b_0$, $0 < \alpha < p - 1$, $0 < \beta < q - 1$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$. We will use the method of sub–supersolutions to establish our results (see e.g. [3] and [4]).

Problem (1.1) arises in the theory of quasi-regular and quasi-conformal mappings (see [17]) as well as in the study of non-Newtonian fluids. In the latter case, the pair $(p, q)$ is a characteristic of the medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with $(p, q) < (2, 2)$ are called pseudoplastics. If $(p, q) = (2, 2)$, they are Newtonian fluids [1].

When $P(x) = Q(x) = a(x) = b(x) = 1$ and $p = q = 2$, systems of the form (1.1) arise in several context in biology and engineering (see [9, 10]). It provides a simple model to describe, for instance, the interaction of three diffusing biological species. See [12] for more details on the physical models involving more general reaction-diffusion system.

On the other hand, the existence of weak solutions for nonlinear elliptic systems involving $p$-Laplacian operators with different weights has been studied using an approximation method (see [6, 7, 15, 16]) and the theory of nonlinear monotone operators method (see [8, 13, 14]).

This paper is organized as follows:

In section 2, we introduce some technical results and notations, which are established in [5]. In section 3, we prove the existence of a positive weak solutions for system (1.1) by using the method of sub–supersolutions. In section 4, we consider the nonexistence results.
2 Technical Results

Now, we introduce some technical results [5] concerning the degenerated homogeneous eigenvalue problem

\[
-\Delta_{R,r} u = -\text{div}[R(x)|\nabla u|^{r-2}\nabla u] = \lambda S(x)|u|^{r-2}u \quad \text{in } \Omega, \\
\Omega, \\
\frac{\nu(x)}{c} < R(x) < c\nu(x), \quad \text{for a.e. } x \in \Omega \text{ with some constant } c \geq 1, \quad (2.2)
\]

and

\[
0 \leq S(x) \in L^{\frac{1}{r-1}}(\Omega) \quad \text{for a.e. } x \in \Omega, \quad (2.4)
\]

with some \( k \) satisfies \( r < k < r^* \) where \( r^* = \frac{N r}{N - r} \), with \( r_s = \frac{r s}{s+1} < r < r^* \) and \( \text{meas } \{ x \in \Omega : S(x) > 0 \} > 0 \). Examples of functions satisfying (2.2) and (2.3) are mentioned in [5].

**Lemma 1** There exists the first eigenvalue \( \lambda^{(r)}_1 > 0 \) and precisely one corresponding eigenfunction \( \phi^{(r)}_{1,r} \geq 0 \) a.e. in \( \Omega \) (\( \phi^{(r)}_{1,r} \) not identical to 0) of the eigenvalue problem (2.1). Moreover, it is characterized by

\[
\lambda^{(r)}_1 \int_{\Omega} S(x)u^r \leq \int_{\Omega} R(x)|\nabla u|^{r}, \quad \text{for all } u \in W^{1,p}_0(P,\Omega) \quad (2.5)
\]

**Lemma 2** Let \( \phi^{(r)}_{1,r} \in W^{1,p}_0(P,\Omega), \phi^{(r)}_{1,r} \geq 0 \) a.e. in \( \Omega \), be the eigenfunction corresponding to the first eigenvalue \( \lambda^{(r)}_1 > 0 \) of the eigenvalue problem (2.1). Then \( \phi^{(r)}_{1,r} \in L^\infty(\Omega) \).

Now, let us introduce the weighted Sobolev space \( W^{1,p}(\nu,\Omega) \) which is the set of all real valued functions \( u \) defined in \( \Omega \) for which (see [5])

\[
\|u\|_{1,r,\nu} = \left[ \int_{\Omega} |u|^r + \int_{\Omega} \nu(x)|\nabla u|^{r} \right]^\frac{1}{r} < \infty. \quad (2.6)
\]
Since we are dealing with the Dirichlet problem, we introduce also the space $W_{0}^{1,r}(\nu, \Omega)$ as the closure of $C_{0}^{\infty}(\Omega)$ in $W^{1,r}(\nu, \Omega)$ with respect to the norm

$$\|u\|_{1,r,\nu} = \left[ \int_{\Omega} \nu(x) |\nabla u|^{r} \right]^{\frac{1}{r}} < \infty,$$

which is equivalent to the norm given by (2.6). Both spaces $W^{1,r}(\nu, \Omega)$ and $W_{0}^{1,r}(\nu, \Omega)$ are well defined reflexive Banach Spaces.

In this paper, we shall take $c = 1$ in (2.2) i.e. $\nu(x) = R(x)$.

### 3 Existence Results

In this section, we shall establish our existence result via the method of sub and supersolutions. A pair of nonnegative functions $(\psi_1, \psi_2), (z_1, z_2) \in W_{0}^{1,p}(P, \Omega) \times W_{0}^{1,q}(Q, \Omega)$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial \Omega$ and

$$\int_{\Omega} P(x) |\nabla \psi_1|^{p-2} \nabla \psi_1 \nabla \zeta dx \leq \lambda \int_{\Omega} a(x) \psi_2^\beta \zeta dx,$$

$$\int_{\Omega} Q(x) |\nabla \psi_2|^{q-2} \nabla \psi_2 \nabla \eta dx \leq \lambda \int_{\Omega} b(x) \psi_1^\alpha \eta dx,$$

$$\int_{\Omega} P(x) |\nabla z_1|^{p-2} \nabla z_1 \nabla \zeta dx \geq \lambda \int_{\Omega} a(x) z_2^\beta \zeta dx,$$

$$\int_{\Omega} Q(x) |\nabla z_2|^{q-2} \nabla z_2 \nabla \eta dx \geq \lambda \int_{\Omega} b(x) z_1^\alpha \eta dx,$$

for all test functions $\zeta \in W_{0}^{1,p}(P, \Omega)$ and $\eta \in W_{0}^{1,q}(Q, \Omega)$ with $\zeta, \eta \geq 0$. Then the following result holds:

**Lemma 3** (see [2, 11]) Suppose there exist sub and supersolutions $(\psi_1, \psi_2)$ and $(z_1, z_2)$ respectively of (1.1) such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then (1.1) has a solution $(u, v)$ such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

Now we shall establish:

**Theorem 4** Let $\vartheta = (p-1)(q-1) - \alpha \beta > 0$. Then system (1.1) has a positive weak solution $(u, v)$ for each $\lambda > 0$. 


Proof of Theorem 4. Let $\lambda_1^{(r)}$ be the first eigenvalue of the eigenvalue problem (2.1) and $\phi_{1,r}$ the corresponding positive eigenfunction satisfying $\phi_{1,r} > 0$ in $\Omega$ and $|\nabla \phi_{1,r}| > 0$ on $\partial \Omega$ with $\|\phi_{1,r}\|_\infty = 1$, for $r = p, q$. Then we have

\begin{equation}
\begin{aligned}
-\Delta_{P,p}\phi_{1,p} &= \lambda_1^{(p)} a(x)|\phi_{1,p}|^{p-2}\phi_{1,p} \quad \text{in } \Omega, \quad \phi_{1,p} = 0 \quad \text{on } \partial \Omega, \\
-\Delta_{Q,q}\phi_{1,q} &= \lambda_1^{(q)} b(x)|\phi_{1,q}|^{q-2}\phi_{1,q} \quad \text{in } \Omega, \quad \phi_{1,q} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\end{equation}

(3.1)

Since $\vartheta = (p-1)(q-1) - \alpha \beta > 0$, we can take $k$ such that

\begin{equation}
\frac{\alpha}{q-1} < k < \frac{p-1}{\beta}.
\end{equation}

(3.2)

We shall verify that $(\psi_1, \psi_2) = (\xi \phi_{1,p}^{p-1}, \xi^k \phi_{1,q}^{q-1})$ is a subsolution of (1.1), where $\xi > 0$ is small and specified later. Let $\zeta \in W_0^{1,p}(P, \Omega)$ with $\zeta \geq 0$. A calculation shows that

\begin{equation}
\begin{aligned}
\int_\Omega P(x)|\nabla \psi_1|^{p-2}\nabla \psi_1 \cdot \nabla \zeta dx &= \left(\frac{p}{p-1}\right)^{p-1} \int_\Omega (P(x)\phi_{1,p}|\nabla \phi_{1,p}|^{p-2}\nabla \phi_{1,p} \cdot \nabla \zeta) dx \\
&= \left(\frac{p}{p-1}\right)^{p-1} \int_\Omega (\lambda_1^{(p)} a(x)\phi_{1,p}^p - P(x)|\nabla \phi_{1,p}|^p) \zeta dx.
\end{aligned}
\end{equation}

Similarly, for $\eta \in W_0^{1,q}(Q, \Omega)$ with $\eta \geq 0$, we have

\begin{equation}
\begin{aligned}
\int_\Omega Q(x)|\nabla \psi_2|^{q-2}\nabla \psi_2 \cdot \nabla \eta dx &= \left(\frac{q}{q-1}\right) \int_\Omega (\lambda_1^{(q)} b(x)\phi_{1,q}^q - Q(x)|\nabla \phi_{1,q}|^q) \eta dx.
\end{aligned}
\end{equation}

Since $\phi_{1,r} = 0$ and $|\nabla \phi_{1,r}| > 0$ on $\partial \Omega$, there is $\delta > 0$ such that

\begin{equation}
\begin{aligned}
\lambda_1^{(p)} a(x)\phi_{1,p}^p - P(x)|\nabla \phi_{1,p}|^p &\leq 0 \quad \text{and} \quad \lambda_1^{(q)} b(x)\phi_{1,q}^q - Q(x)|\nabla \phi_{1,q}|^q &\leq 0 \quad \text{on } \overline{\Omega}_\delta
\end{aligned}
\end{equation}

with $\overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial \Omega) \leq \delta\}$. This shows that

\begin{equation}
\begin{aligned}
\left(\frac{p}{p-1}\right)^{p-1} \int_{\Omega_\delta} (\lambda_1^{(p)} a(x)\phi_{1,p}^p - P(x)|\nabla \phi_{1,p}|^p) \zeta dx &\leq \lambda \int_{\Omega_\delta} a(x)\psi_2^\beta \zeta dx,
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\left(\frac{q}{q-1}\right) \int_{\Omega_\delta} (\lambda_1^{(q)} b(x)\phi_{1,q}^q - Q(x)|\nabla \phi_{1,q}|^q) \eta dx &\leq \lambda \int_{\Omega_\delta} b(x)\psi_1^\eta \eta dx.
\end{aligned}
\end{equation}
Furthermore, we note that \( \phi_{1,p}, \phi_{1,q} \geq \sigma \) in \( \Omega - \overline{\Omega}_\delta \) for some \( \sigma > 0 \). Then from (3.2) there is \( \xi_0 > 0 \) such that if \( \xi \in (0, \xi_0) \), the following inequalities hold:

\[
\xi^{p-1-k\beta} \left( \frac{p}{p-1} \right)^{p-1} \lambda_1^{(p)} \phi_{1,p}^p \leq \lambda \sigma^{\frac{\alpha p}{q-1}} \leq \lambda \phi_{1,q}^{\frac{\alpha p}{q-1}} \quad \text{in} \quad \Omega - \overline{\Omega}_\delta,
\]

and

\[
\xi^{k(q-1)-\alpha} \left( \frac{q}{q-1} \right)^{q-1} \lambda_1^{(q)} \phi_{1,q}^q \leq \lambda \sigma^{\frac{\alpha q}{p-1}} \leq \lambda \phi_{1,p}^{\frac{\alpha q}{p-1}} \quad \text{in} \quad \Omega - \overline{\Omega}_\delta.
\]

Then, we have

\[
\int_{\Omega - \overline{\Omega}_\delta} P(x)|\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla \zeta \, dx = \left( \frac{p}{p-1} \right)^{p-1} \xi^{p-1} \int_{\Omega - \overline{\Omega}_\delta} (\lambda_1^{(p)} a(x) \phi_{1,p}^p - P(x)|\nabla \phi_{1,p}|^p) \zeta \, dx
\]

\[
\leq \lambda \int_{\Omega - \overline{\Omega}_\delta} a(x) \xi^{\beta k \phi_{1,q}^{\frac{\alpha p}{q-1}}} \zeta \, dx = \lambda \int_{\Omega - \overline{\Omega}_\delta} a(x) \psi_2^{\beta} \zeta \, dx.
\]

Similarly,

\[
\int_{\Omega - \overline{\Omega}_\delta} Q(x)|\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla \eta \, dx = \left( \frac{q}{q-1} \right)^{q-1} \xi^{q-1} \int_{\Omega - \overline{\Omega}_\delta} (\lambda_1^{(q)} b(x) \phi_{1,q}^q - Q(x)|\nabla \phi_{1,q}|^q) \eta \, dx
\]

\[
\leq \lambda \int_{\Omega - \overline{\Omega}_\delta} b(x) \xi^{\alpha \phi_{1,p}^{\frac{\alpha q}{p-1}}} \eta \, dx = \lambda \int_{\Omega - \overline{\Omega}_\delta} b(x) \psi_1^{\alpha} \eta \, dx,
\]

i.e. \( (\psi_1, \psi_2) \) is a subsolution of (1.1).

Next, we construct a supersolution \((z_1, z_2)\) of (1.1). Let \( e_r(x) \) be the positive solution of (see [16])

\[
-\Delta_{R,r} e_r = 1 \quad \text{in} \quad \Omega, \quad e_r = 0 \quad \text{on} \quad \partial \Omega \quad \text{for} \quad r = p, q.
\]

We denote \( z_1(x) = Ae_p, \quad z_2(x) = Be_q, \) where the constants \( A, B > 0 \) are large and to be chosen later. We shall verify that \((z_1, z_2)\) is the supersolution of (1.1). To do this, let \( \zeta \in W_{0}^{1,p}(\Omega) \) with \( \zeta \geq 0 \). Then we have

\[
\int_{\Omega} P(x)|\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla \zeta \, dx = A^{p-1} \int_{\Omega} P(x)|\nabla e_p|^{p-2} \nabla e_p \cdot \nabla \zeta \, dx = A^{p-1} \int_{\Omega} \zeta \, dx.
\]

Similarly, for \( \eta \in W_{0}^{1,q}(\Omega, \Omega) \) with \( \eta \geq 0 \), we have

\[
\int_{\Omega} Q(x)|\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla \eta \, dx = B^{q-1} \int_{\Omega} Q(x)|\nabla e_q|^{q-2} \nabla e_q \cdot \nabla \eta \, dx = B^{q-1} \int_{\Omega} \eta \, dx.
\]
Since \( \psi > 0 \), it is easy to prove that there exist positive large constants \( A, B \) such that

\[
A^{p-1} = \lambda a B^\beta \mu^\beta_q, \quad B^{q-1} = \lambda b A^\alpha \mu^\alpha_p,
\]

where \( \mu_r = \|e_r\|_\infty ; r = p, q \). These imply that

\[
\int_{\Omega} P(x)|\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla \zeta dx = \lambda \int_{\Omega} l_a B^\beta \mu^\beta_q \geq \lambda \int_{\Omega} a(x) z_2^\beta \zeta dx,
\]

and

\[
\int_{\Omega} Q(x)|\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla \eta dx = B^{q-1} \int_{\Omega} \eta dx = \lambda \int_{\Omega} l_b A^\alpha \mu^\alpha_p \geq \lambda \int_{\Omega} b(x) z_1^\alpha \eta dx,
\]

i.e. \((z_1, z_2)\) is a supersolution of (1.1) with \( z_i \geq \psi_i \) with large \( A, B \), for \( i = 1, 2 \). Thus, there exists a solution \((u, v)\) of (1.1 ) with \( \psi_1 \leq u \leq z_1, \psi_2 \leq v \leq z_2 \). This completes the proof of Theorem 4.

### 4 Nonexistence result

In this section, under some conditions we prove that system (1.1) has no positive weak solution.

**Theorem 5** Suppose that \( \vartheta = (p - 1)(q - 1) - \alpha \beta = 0, p \beta = q(p - 1) \) and \( a(x) = b(x) \). Then there exists \( \lambda_0 > 0 \) such that for \( 0 < \lambda < \lambda_0 \), system (1.1) has no positive weak solution.

**Proof.** Let us assume that \((u, v) \in W_0^{1,p}(P, \Omega) \times W_0^{1,p}(Q, \Omega)\) be a positive weak solution of (1.1). We prove Theorem 4 by arriving at a contradiction.

Multiplying the first equation of (1.1) by \( u \), we have from Young inequality that

\[
\int_{\Omega} P(x)|\nabla u|^{p-2} dx \leq \lambda \int_{\Omega} a(x) \left( \frac{u^p}{\mu_1} + \frac{v^q}{\mu_2} \right) dx,
\]

with \( \mu_1 = p > 1 \) and \( \mu_2 = \frac{q}{q-1} > 1 \).

Similarly, we have

\[
\int_{\Omega} Q(x)|\nabla v|^{q-2} dx \leq \lambda \int_{\Omega} b(x) \left( \frac{u^p}{\theta_1} + \frac{v^q}{\theta_2} \right) dx,
\]
with $\theta_1 = \frac{q}{q-1} > 1$ and $\theta_2 = q > 1$.

Note that

$$
\lambda_{1}^{(p)} \int_{\Omega} a(x)u^{p}dx \leq \int_{\Omega} P(x)|\nabla u|^{p}dx, \quad \lambda_{1}^{(q)} \int_{\Omega} b(x)v^{q}dx \leq \int_{\Omega} Q(x)|\nabla v|^{q}dx.
$$

Combining (4.1)-(4.3), we obtain

$$
\lambda_{1}^{(p)} \int_{\Omega} a(x)u^{p}dx + \lambda_{1}^{(q)} \int_{\Omega} b(x)v^{q}dx \leq \lambda \int_{\Omega} \left( \frac{a(x)}{\mu_1} + \frac{b(x)}{\theta_1} \right)u^{p}dx + \lambda \int_{\Omega} \left( \frac{a(x)}{\mu_2} + \frac{b(x)}{\theta_2} \right)v^{q}dx.
$$

Now, if $a(x) = b(x)$, we have

$$
(\lambda_{1}^{(p)} - \lambda) \int_{\Omega} a(x)u^{p}dx + (\lambda_{1}^{(q)} - \lambda) \int_{\Omega} a(x)v^{q}dx \leq 0,
$$

which is a contradiction if $0 < \lambda < \lambda_0 = \min\{\lambda_{1}^{(p)}, \lambda_{1}^{(q)}\}$. Thus system (1.1) has no positive weak solution if $a(x) = b(x)$ and $\lambda \in (0, \lambda_0)$.

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