Travelling Wave Solutions of the Zakharov-Kuznetsov Equation in Plasmas with Power Law Nonlinearity

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Abstract

In this paper, the Cole-Hopf transformation and the first integral method were used to construct traveling wave solutions for the Zakharov-Kuznetsov equation in plasmas with power law nonlinearity. The traveling wave solutions are expressed by the complex hyperbolic functions, the complex trigonometric functions, and the complex rational functions. The first integral method presents a wider applicability for handling nonlinear evolution equations.

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1. Introduction

The investigation of the traveling wave solutions for nonlinear partial differential equations (NPDEs) plays an important role in the study of nonlinear physical phenomena. There has been a lot of equations that has been studied in detail in this area including the nonlinear Schrödinger equation that appears in the context of Plasma Physics, Nonlinear Optics, Fluid Dynamics and other areas as well [1,2]. In this paper, one such nonlinear evolution equation will be studied. This is known as the Zakharov-Kuznetsov equation (ZKE). This ZKE appears in many areas of Physics, Applied Mathematics, and Engineering. In particular, it shows up in the area of Plasma Physics [3,4,5].

The ZKE, which is an isotropic 2+1-dimensional nonlinear evolution equation, was first derived for weakly nonlinear ion-acoustic waves in a plasma comprising of cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1,2,5].

The ZKE, which is an isotropic 2+1-dimensional nonlinear evolution equation, was first derived for weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [5].

A variety of powerful methods such as inverse scattering method [6,7], the tanh–sech method [8-10], extended tanh method [11-13], sine–cosine method [14,15], homogeneous balance method [16], Exp-function method [17-19], were used to develop nonlinear problems.

The pioneer work Feng [20] introduced the first integral method for a reliable treatment of NPDEs. The useful first interest integral method is widely used by many such as in [21-24] and the references therein.

In Section 2, we describe this method for finding exact traveling wave solutions of nonlinear evolution equations (NEEs). In Section 4, we use the Cole–Hopf transformation and the first integral method in detail with the Zakharov–Kuznetsov equation in plasmas with power law nonlinearity. In Section 5, some conclusions are given.

2. The first integral method [22]

Consider a general NPDE in the form

\[ P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxt}, \ldots) = 0. \]  (1)

Using the wave variable \( \xi = x - ct \) carries (1) into the following ordinary differential equation (ODE)
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\[ Q(U, U', U'', \ldots) = 0 \]  \hspace{1cm} (2)

where prime denotes the derivative with respect to the same variable \( \xi \).

Next, we introduce new independent variables \( x = u \), \( y = u_\xi \) which change (2) to a system of ODEs

\[
\begin{align*}
  x' &= y \\
  y' &= f(x, y)
\end{align*}
\]  \hspace{1cm} (3)

According to the qualitative theory of differential equations [25], if one can find the first integrals to System(3) under the same conditions, the analytic solutions to (3) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way. A key idea of this approach here to find the first integral is to utilize the Division Theorem. For convenience, first let us recall the Division Theorem for two variables in the complex domain \( C \).

**Division Theorem.** Suppose that \( P(x, y) \) and \( Q(x, y) \) are polynomials of two variables \( x \) and \( y \) in \( C[x, y] \) and \( P(x, y) \) is irreducible in \( C[x, y] \). If \( Q(x, y) \) vanishes at all zero points of \( P(x, y) \), then there exists a polynomial \( G(x, y) \) in \( C[x, y] \) such that \( Q(x, y) = P(x, y) \cdot G(x, y) \).

The Division Theorem follows immediately from the Hilbert – Nullstellensatz Theorem [27]:

**Hilbert – Nullstellensatz Theorem.** Let \( k \) be a field and \( L \) an algebraic closure of \( k \).

(i) Every ideal \( \mathcal{I} \) of \( k[X_1, \ldots, X_n] \) not containing 1 admits at least one zero in \( L^n \).

(ii) Let \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) be two elements of \( L^n \); for the set of polynomials of \( k[X_1, \ldots, X_n] \) zero at \( x \) to be identical with the set of...
polynomials of \( k[X_1, ..., X_n] \) zero at \( y \), it is necessary and sufficient that there exists a \( k \)-automorphism \( s \) of \( L \) such that \( y_i = s(x_i) \) for \( 1 \leq i \leq n \).

(iii) For an ideal \( \alpha \) of \( k[X_1, ..., X_n] \) to be maximal, it is necessary and sufficient that there exists an \( x \) in \( L^n \) such that \( \alpha \) is the set of polynomials of \( k[X_1, ..., X_n] \) zero at \( x \).

(iv) For a polynomial \( Q \) of \( k[X_1, ..., X_n] \) to be zero on the set of zeros in \( L^n \) of an ideal \( \gamma \) of \( k[X_1, ..., X_n] \), it is necessary and sufficient that there exists an integer \( m > 0 \) such that \( Q^m \in \gamma \).

3. The ZKE with power law nonlinearity

The dimensionless form of the ZKE, with power law nonlinearity, that is going to be studied in this paper is given by [28]

\[
 u_t + a \, u^n \, u_x + b \, (u_{xx} + u_{yy})_x = 0 \tag{4}
\]

In (4), \( a \) and \( b \) are real valued constants. The first term represents the evolution term while the second term is the nonlinear term and finally the third and fourth terms together, in parentheses, are the dispersion terms. The solitons are a result of a delicate balance between dispersion and nonlinearity. The exponent \( n \), which indicates the power law, is a positive real number. The special case where \( n = \frac{1}{2} \) gives the modified ZKE.

In this paper, the traveling wave solutions to Eq.(4) will be obtained, for any general \( n \), by using the Cole-Hopf transformation and the first integral method. While Eq.(4) has been solved previously by the extended hyperbolic function method, the solutions that will be derived here were have a generalized structure as follows:

Making the Cole-Hopf transformation

\[
 U = u^k \quad , \quad k \in \mathbb{R} \quad , \quad k \neq 0
\]

, and assuming that the resulting equation has traveling wave solutions as
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\[ U(x, y, t) = U(\xi) \quad , \quad \xi = x + y - ct \ , \text{ where } c \in \mathfrak{R} \text{ is wave velocity,} \]

then we have

\[
\left( -\frac{cn}{2bk} \right) U + \left( \frac{k}{n} - 1 \right) \frac{U'^2}{U} + U'' + \left( \frac{an}{2bk(n+1)} \right) U^{k+1} = 0, \tag{5}
\]

where \( U', U'' \) denote \( \frac{dU}{d\xi} \) and \( \frac{d^2U}{d\xi^2} \), respectively.

Considering the case of \( k = 2 \) in (5), i.e.

\[
\left( -\frac{cn}{4b} \right) U + \left( \frac{2}{n} - 1 \right) \frac{U'^2}{U} + U'' + \left( \frac{an}{4b(n+1)} \right) U^3 = 0 . \tag{6}
\]

If we let \( x = U \), \( y = \frac{dU}{d\xi} \), the Eq.(6) is equivalent to the two-dimensional autonomous system

\[
\begin{align*}
\frac{dx}{d\xi} &= y \\
\frac{dy}{d\xi} &= \left( \frac{cn}{4b} \right) x - \left( \frac{an}{4b(n+1)} \right) x^3 + \left( 1 - \frac{2}{n} \right) \frac{y^2}{x} 
\end{align*} \tag{7}
\]

Assume that

\[ d\tau = \frac{d\xi}{x} \]

, then (7) becomes

\[
\begin{align*}
\frac{dx}{d\tau} &= xy \\
\frac{dy}{d\tau} &= \left( \frac{cn}{4b} \right) x^2 - \left( \frac{an}{4b(n+1)} \right) x^4 + \left( 1 - \frac{2}{n} \right) y^2 
\end{align*} \tag{8}
\]
Now, we are applying the Division Theorem to seek the first integral to system (8). Suppose that \( x = x(\tau), \ y = y(\tau) \) are the nontrivial solutions to (8), and

\[
p(x,y) = \sum_{i=0}^{m} a_i(x) y^i,\text{ is an irreducible polynomial in } C[x,y,]\text{, such that}
\]

\[
p[x(\tau), y(\tau)] = \sum_{i=0}^{m} a_i(x(\tau)) y(\tau)^i = 0, \quad (9),
\]

where \( a_i(x) \) \( (i=0,1,\ldots,m) \) are polynomials of \( x \) and \( a_m \neq 0 \).

We call (9) the first integral of polynomial form to system (8). We start our study by assuming \( m = 1 \) in (9). Note that \( \frac{dp}{d\tau} \) is a polynomial in \( x \) and \( y \), and

\[
p[x(\tau), y(\tau)] = 0 \text{ implies } \frac{dp}{d\tau} \mid_{(8)} = 0. \text{ According to the Division Theorem, there exists a polynomial } H(x,y) = h(x) + g(x)y \text{ in } C[x,y] \text{ such that}
\]

\[
\frac{dp}{d\tau} \mid_{(8)} = \left[ \frac{\partial p}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial p}{\partial y} \frac{\partial y}{\partial \tau} \right] \mid_{(8)},
\]

\[
= \sum_{i=0}^{1} \left( a_i'(x) y^{i-1} \right) + \sum_{i=0}^{1} \left( i a_i(x) y^{i-1} \right) \left( \frac{c n}{4 b} \right) x^2 - \left( \frac{a n}{4b(n+1)} \right) x^4 + \left( 1 - \frac{2}{n} \right) y^2, \quad (10)
\]

,where prime denotes differentiation with respect to the variable \( x \).

On equating the coefficients of \( y^i \) \( (i = 2,1,0) \) on both sides of (10), we have

\[
x a_1'(x) + \left( 1 - \frac{2}{n} \right) a_1(x) = g(x) a_1(x), \quad (11)
\]

\[
x a_0'(x) = h(x) a_1(x) + g(x) a_0(x), \quad (12)
\]

\[
a_1(x) \left[ \left( \frac{c n}{4 b} \right) x^2 - \left( \frac{a n}{4b(n+1)} \right) x^4 \right] = h(x) a_0(x). \quad (13)
\]

Since \( a_1(x) \) is a polynomial of \( x \), from (11) we conclude that \( a_1(x) \) is a constant and \( g(x) = 1 - \frac{2}{n} \). For simplicity, we take \( a_1(x) = 1 \), and balancing the degrees of \( h(x) \) and \( a_0(x) \) we conclude that \( \text{deg}(h(x)) = 2 \) and \( \text{deg}(a_0(x)) = 2 \), only. Now suppose that
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\[ h(x) = A_2 x^2 + A_1 x + A_0, a_0(x) = B_2 x^2 + B_1 x + B_0 \quad (A_2 \neq 0, B_2 \neq 0), \quad (14) \]

where \( A_2, A_1, A_0, B_2, B_1, B_0 \) are all constants to be determined. Substituting Eq.(14) into Eq.(12), we obtain

\[ h(x) = \left(1 + \frac{2}{n} B_2 \right) x^2 + \left(\frac{2B_1}{n}\right) x - B_0 \left(1 - \frac{2}{n}\right). \]

Substituting \( a_0(x), a_1(x) \) and \( h(x) \) in (13), and setting all the coefficients of powers \( x \) to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain the following solutions

\[ c = -\frac{2i \sqrt{a} \sqrt{b}}{\sqrt{1+n} \sqrt{2+n}} n B_0, \quad B_1 = 0, \quad B_2 = -\frac{i \sqrt{a} n}{2 \sqrt{a} \sqrt{1+n} \sqrt{2+n}}, \quad (15) \]

\[ c = \frac{2i \sqrt{a} \sqrt{b}}{\sqrt{1+n} \sqrt{2+n}} n B_0, \quad B_1 = 0, \quad B_2 = \frac{i \sqrt{a} n}{2 \sqrt{a} \sqrt{1+n} \sqrt{2+n}}, \quad (16) \]

Setting (15) and (16) in (9), we obtain the system (8) has one first integral

\[ y \pm \left(\frac{i \sqrt{a} n}{2 \sqrt{b} \sqrt{1+n} \sqrt{1+n}}\right) x^2 + B_0 = 0. \]

respectively. Combining this first integral with (8), the second – order differential Eq.(6) can be reduced to

\[ \frac{dU}{d\xi} = \pm \left(\frac{i \sqrt{a} n}{2 \sqrt{b} \sqrt{1+n} \sqrt{1+n}}\right) U^2 - B_0. \quad (17) \]

Solving Eq.(17) directly and changing to the original variables, we obtain the following complex traveling solitary wave solution and the complex periodic wave solution to Eq.(4):
\[ u_1(x, y, t) = K \tanh \left( \frac{1}{2} \sqrt{n} (x + y) + \frac{2i \sqrt{a} \sqrt{b} n B_0}{\sqrt{1+n} \sqrt{2+n}} (1 - 2 \sqrt{b} \sqrt{1+n} \sqrt{2+n} c_1) \sqrt{B_0} \right) \]

\[ u_2(x, y, t) = \frac{1}{2} \sqrt{n} (x + y) - \frac{2i \sqrt{a} \sqrt{b} n B_0}{\sqrt{1+n} \sqrt{2+n}} (1 - 2 \sqrt{b} \sqrt{1+n} \sqrt{2+n} c_1) \sqrt{B_0} \]

where \( K = - \frac{1}{a b^{1/4} \sqrt{n}} (1 - i) b^{1/4} (1+n)^{-1/4} (2+n)^{-1/4} \sqrt{B_0} \)

and \( c_1 \) is an arbitrary constant, respectively.

Now we assume that \( m = 2 \) in (9).

By the Division Theorem, there exists a polynomial \( H(x,y) = h(x) + g(x) y \) in \( C[x,y] \) such that

\[
\frac{dp}{d\tau} \bigg |_{(8)} = (\frac{\partial p}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial p}{\partial y} \frac{\partial}{\partial \tau}) \bigg |_{(8)},
\]

\[
= \sum_{i=0}^{2} (a_i'(x) y^i \cdot x y) + \sum_{i=0}^{2} \left( i a_i(x) y^{i-1} \left[ \frac{c n}{4 b} \right] x^2 - \left( \frac{a n}{4 b (n+1)} \right) x^4 + \left( 1 - \frac{2}{n} \right) y^2 \right),
\]

\[
= (h(x) + g(x) y) \left( \sum_{i=0}^{2} a_i(x) y^i \right),
\]

(18)

On equating the coefficients of \( y^i \) \( (i = 3,2,1,0) \) on both sides of (18), we have

\[
x a_2'(x) + 2 \left( 1 - \frac{2}{n} \right) a_2(x) = g(x) a_2(x),
\]

(19)

\[
x a_1'(x) + (1 - \frac{2}{n}) a_1(x) = h(x) a_2(x) + g(x) a_1(x).
\]

(20)

\[
x a_0'(x) + 2 a_2(x) \left[ \frac{c n}{4 b} \right] x^2 - \left( \frac{a n}{4 b (n+1)} \right) x^4 = h(x) a_1(x) + g(x) a_0(x),
\]

(21)

\[
a_1(x) \left[ \frac{c n}{4 b} \right] x^2 - \left( \frac{a n}{4 b (n+1)} \right) x^4 = h(x) a_0(x).
\]

(22)
Since, \( a_2(x) \) is a polynomial of \( x \), from (19) we conclude that \( a_2(x) \) is a constant and \( g(x) = 2 \left( 1 - \frac{2}{n} \right) \). For simplicity, we take \( a_2(x) = 1 \), and balancing the degrees of \( h(x) \), \( a_0(x) \) and \( a_1(x) \) we conclude that \( \deg(h(x)) = 1 \), \( \deg(a_1(x)) = 1 \); \( \deg(h(x)) = 2 \), \( \deg(a_1(x)) = 2 \).

**Case 1.** \( \deg(h(x)) = 1 \) and \( \deg(a_1(x)) = 1 \).

In this case, we assume that

\[
 h(x) = A_1 \, x + A_0 \quad , \quad a_1(x) = B_1 \, x + B_0 ,
\]

where \( A_1, A_0, B_1, B_0 \) are constants to be determined.

Inserting Eq.(23) into Eqs.(20) and (21), we deduce that

\[
 h(x) = \left( 2 \frac{B_1}{n} \right) x - B_0 \left( 1 - \frac{2}{n} \right) ;
\]

\[
 a_0(x) = \left( \frac{a}{4 \, b \, (1 + n) \, (2 + n)} \right) x^4 - \left( \frac{c \, n^2}{8 \, b} \right) x^2 + \frac{1}{2} \left( B_0 + x \, B_1 \right)^2 + F \, x^{2 - \frac{4}{n}} ,
\]

where \( F \) is an arbitrary integration constant.

Substituting \( a_0(x) \), \( a_1(x) \) and \( h(x) \) in (22), and setting all the coefficients of powers \( x \) to be zero, we obtain a system of nonlinear algebraic equations and by solving it we get

\[
 F = 0 \quad , \quad c = 0 \quad , \quad B_0 = 0 . \quad (24)
\]

Setting (24) in (9), we get system (8) has the first integral

\[
 y_1(x) = - \frac{x \, B_1}{2} - \frac{\sqrt{-b \, (1+n) \, (2+n) \, x^2 \left( a \, n^2 \, x^2 + b \, (1+n) \, (2+n) \, B_1^2 \right)}}{2 \, b \, (1+n) \, (2+n)} , \quad (25)
\]

\[
 y_2(x) = - \frac{x \, B_1}{2} + \frac{\sqrt{-b \, (1+n) \, (2+n) \, x^2 \left( a \, n^2 \, x^2 + b \, (1+n) \, (2+n) \, B_1^2 \right)}}{2 \, b \, (1+n) \, (2+n)} , \quad (26)
\]

Unfortunately, it is not convenient to derive the explicit solutions for the Eq. (4) from Eqs. (25) and (26), so we omit these implicit solutions in such cases.
Case 2. \( \deg(h(x)) = 2 \) and \( \deg(a_1(x)) = 2 \).

In this case, we assume that

\[
h(x) = A_2 x^2 + A_1 x + A_0, \quad a_1(x) = B_2 x^2 + B_1 x + B_0, \quad (A_2 \neq 0, \ B_2 \neq 0),
\]

where \( A_2, A_1, A_0, B_2, B_1, B_0 \) are constants to be determined.

Substituting Eq. (27) into Eqs. (20) and (21), we have

\[
h(x) = \left(1 + \frac{2}{n} \right) B_2 x^2 + \frac{2 B_1}{n} x - B_0 \left(1 - \frac{2}{n} \right);
\]

\[
a_0(x) = \left( \frac{a n^2}{4b(1+n)(2+n)} + \frac{B_2^2}{2} \right) x^4 + B_1 B_2 x^3 + \left( B_0 B_2 + \frac{B_1^2}{2} - \frac{c n^2}{8 b} \right) x^2 + B_0 B_1 x
\]

\[
+ \frac{B_0^2}{2} + F x^{2-4/n},
\]

where \( F \) is an arbitrary integration constant.

Substituting \( a_0(x), a_1(x) \) and \( h(x) \) in (22), and setting all the coefficients of powers \( x \) to be zero, we obtain a system of nonlinear algebraic equations and by solving it we obtain

\[
F = 0, \quad c = 0, \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = -\frac{i \sqrt{a} n}{\sqrt{b} \sqrt{1+n} \sqrt{2+n}},
\]

\[
F = 0, \quad c = 0, \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = \frac{i \sqrt{a} n}{\sqrt{b} \sqrt{1+n} \sqrt{2+n}},
\]

Setting (28) and (29) in (9) we obtain

\[
y \mp \left( \frac{i \sqrt{a} n}{2 \sqrt{b} \sqrt{1+n} \sqrt{2+n}} \right) x^2 = 0
\]

Using this first integral, the second – order ODE (6) reduces to

\[
\frac{dU}{d\xi} = \pm \left( \frac{i \sqrt{a} n}{2 \sqrt{b} \sqrt{1+n} \sqrt{2+n}} \right) U^2.
\]
Solving Eq. (30) and changing to the original variables, we obtain the complex rational function solutions to Eq. (4)

\[
\begin{align*}
\frac{u_3(x, y, t)}{2^n} & = \left[ -i \sqrt{a} n (x + y) - 2 \sqrt{b} \sqrt{1 + n} \sqrt{2 + n} c_1 \right] \\
\frac{u_4(x, y, t)}{2^n} & = \left[ i \sqrt{a} n (x + y) - 2 \sqrt{b} \sqrt{1 + n} \sqrt{2 + n} c_1 \right]
\end{align*}
\]

where \( c_1 \) is an arbitrary constant.

These solutions are all new exact solutions.

5. Conclusion

Using the Cole-Hopf transformation and the first integral technique, we have obtained new solutions of the ZKE with power law nonlinearity. These solutions include complex traveling solitary wave solutions, complex periodic wave solutions, some solutions are complex rational functions and some implicit solutions. From these results, we can see that the first integral method is one of the most effective methods to obtain traveling wave solutions, especially for NEEs.

References


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