Common Fixed Point Theorem for Occasionally Weakly Compatible Mapping in Q-Fuzzy Metric Spaces Satisfying Integral type Inequality

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Abstract
This paper presents some common fixed point theorem for Occasionally Weakly Compatible mapping in Q-fuzzy metric spaces satisfying integral type inequality.

Keywords: Contractive condition of integral type, Fixed point, Occasionally Weakly Compatible mapping, Q-fuzzy metric spaces, t-norm

1. Introduction


The Q-fuzzy metrics spaces is introduced by Guangpeng Sun and Kai Yang[7] which can be considered as a Generalization of fuzzy metric spaces. Sessa [18] improved commutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space.

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R. Vasuki [14] proved fixed point theorems for R-weakly commuting mapping Pant [15,16,17] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. The concept of compatible maps by [10] and weakly compatible maps by [8] in fuzzy metric space is generalized by A. Al Thagafi and Naseer Shahzad [1] by introducing the concept of occasionally weakly compatible mappings. Recent results on fixed point in Q-fuzzy metric space can be viewed in [7]. In this paper we have improved the result of Guangpeng Sun and Kai Yang [7] by proving the same theorem with more weaker condition occasionally weakly compatible mappings in Q-fuzzy metric spaces satisfying integral type inequality.

2. Preliminary Notes

**Definition 2.1[2]** A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if it satisfy the following condition:

(i) $*$ is associative and commutative.
(ii) $*$ is continuous function.
(iii) $a * 1 = a$ for all $a \in [0,1]$.
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

**Definition 2.2[7]** A 3-tuple $(X, Q, *)$ is called a Q-fuzzy metric space if $X$ is an arbitrary (non-empty) set $*$ is a continuous t-norm, and $Q$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$:

(i) $Q(x, x, y, t) > 0$ and $Q(x, x, y, t) \leq Q(x, y, z, t)$ for all $x, y, z \in X$ with $z = y$.
(ii) $Q(x, y, z, t) = 1$ if and only if $x = y = z$.
(iii) $Q(x, y, z, t) = Q(p(x, y, z), t)$ (symmetry) where $p$ is a permutation function.
(iv) $Q(x, a, a, t) * Q(a, y, z, s) \leq Q(x, y, z, t + s)$.
(v) $Q(x, y, z, t)(0, \infty) \rightarrow [0,1]$ is continuous.

A Q-fuzzy metric space is said to be symmetric if $Q(x, y, y, t) = Q(x, x, y, t)$ for all $x, y \in X$.

**Definition 2.3[6]** Let $(X, Q, *)$ be a Q-fuzzy metric space. A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $Q(x_m, x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $Q(x_m, x_n, x, t) > 1 - \varepsilon$ for each $l, n, m \geq n_0$. 


Definition 2.4 [6]: The Q-fuzzy metric space is called to be complete if every Cauchy sequence is convergent, the sequence \( \{x_n\} \) in \( X \) also converges to \( x \) if and only if \( Q(x_{n},x_{n},x,t) \to 1 \) as \( n \to \infty \), for each \( t > 0 \) and it is a Cauchy sequence if for each \( 0 < \varepsilon < 1 \) and \( t > 0 \), there exist \( n_0 \in \mathbb{N} \) such that \( Q(x_{m},x_{n},x_{n}) > 1 - \varepsilon \) for each \( n,m \geq n_0 \).

Lemma 2.5 [7]: If \( (X,Q,*) \) be a Q-fuzzy metric space, then \( Q(x,y,z,t) \) is non-decreasing with respect to \( t \) for all \( x,y,z \) in \( X \).

Proof: Proof is this is implicated in [7].

Lemma 2.6 [7]: Let \( (X,Q,*) \) be a Q-fuzzy metric space.
(a) If there exists a positive number \( k < 1 \) such that:
\[
Q(y_{n+2},y_{n+1},y_{n+1},kt) \geq Q(y_{n+1},y_{n},y_{n},t), t > 0, n \in \mathbb{N}
\]
then \( \{y_n\} \) is a Cauchy sequence in \( X \).
(b) If there exists \( k \in (0,1) \) such that \( Q(x,y,y,kt) \geq Q(x,y,y,t) \) for all \( x,y \in X \) and \( t > 0 \) then \( x = y \).

Proof: By the assume \( \lim_{n \to \infty} Q(x,y,z,t) = 1 \) and the property of non-decreasing, it is easy to get the results.

Definition 2.7 [3]: Let \( X \) be a set, \( f \) and \( g \) selfmaps of \( X \). A point \( x \in X \) is called a coincidence point of \( f \) and \( g \) iff \( fx = gx \). We shall call \( w = fx = gx \) a point of coincidence of \( f \) and \( g \).

Definition 2.8 [7]: Let \( f \) and \( g \) be self maps on a Q-fuzzy metric space \( (X,Q,*) \). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, \( fx = gx \) implies that \( fgx = gfx \).

Definition 2.9 [7]: Let \( f \) and \( g \) be self maps on a Q-fuzzy metric space \( (X,Q,*) \). The pair \( (f,g) \) is said to be compatible if \( \lim_{n \to \infty} Q(fgx_{n},gfx_{n},gx_{n},t) = 1 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_{n} = \lim_{n \to \infty} gfx_{n} = z \) for some \( z \in X \).

Definition 2.10 [1]: Two self maps \( f \) and \( g \) of a set \( X \) are occasionally weakly compatible (owc) iff there is a point \( x \) in \( X \) which is coincidence point of \( f \) and \( g \) at which \( f \) and \( g \) commute. Al-ThagaW and Naseer [1](2008) shown that occasionally weakly is weakly compatible but converse is not true.

Example: Let \( R \) be the usual metric space. Define \( S,T : R \to R \) by \( Sx = x \) and \( Tx = x^3 \) for all \( x \in R \). Then \( Sx = Tx \) for \( x = 0, 2 \) but \( ST0 = TS0 = ST2 \neq T2 \).
TS2. S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.11 [1]: Let X be a set, f, g owc self maps of X. If f and g have unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

3. Main Result

Theorem 3.1: Let f, T, g and S a self mapping of a complete symmetric Q-fuzzy metric space with continuous t-norm satisfying the following condition
(i) The pair {f, T} and {g, S} be owc.
(ii) For all x, y in X, k in (0,1), t > 0 such that
\[ \int_0^t \phi(t) dt \leq \int_0^t Q(fx, gy, sy, yt) + Q(gy, sy, sy, yt) + Q(fx, sy, yt) \phi(t) dt \]
where \( \phi(t): R^+ \rightarrow R \) is a Lebesgue-integrable mapping which is summable, nonegative and such that \( \int_0^e \phi(t) dt > 0 \) for each \( e > 0 \) then there exist a unique point \( w \in X \) such that \( f w = Tw = w \) and a unique point \( z \in X \) such that \( g z = Sz = z \). Moreover, \( z = w \) so that there is a unique common fixed point of f, g, S and T.

Proof: Let the pair \{f, T\} and \{g, S\} be owc, so there are point \( x, y \in X \) such that \( fx = Tx \) and \( gy = Sy \). We claim that \( fx = gy \). If not by inequality (ii)
\[ \int_0^t \phi(t) dt \geq \int_0^t Q(fx, gy, gy, yt) + Q(gy, gy, gy, yt) + Q(fx, gy, gt) \phi(t) dt \]
\[ \geq \int_0^t Q(fx, gy, yt) + Q(gy, gy, yt) \phi(t) dt \]
\[ \geq \int_0^t Q(fx, gy, yt) \phi(t) dt \]
Therefore \( fx = gy \) i.e. \( fx = Tx = gy = Sy \).

Suppose that there is another point \( z \) such that \( f z = T z = z \) then by (1) we have \( f z = T z = g y = Sy \). So \( fx = fz \) and \( w = fx = Tx \) is the unique point of coincidence of f and g by Lemma 2.11 w is the only common fixed point of f and g. Similarly there is a unique point \( z \in X \) such that \( gz = Sz \).

Assume that \( w \neq z \). We have
\[ \int_0^t Q(w, z, z, rt) \phi(t) dt \geq \int_0^t Q(fw, gz, Sz, rt) \phi(t) dt \]
\[ \geq \int_0^t Q(Tw, Sz, Sz, rt) + Q(gz, Sz, Sz, rt) + Q(fw, Sz, Sz, rt) \phi(t) dt \]
\[ = \int_0^t Q(w, z, z, rt) + Q(z, z, z, rt) + Q(w, zz, rt) \phi(t) dt \]
\[ \geq \int_0^t Q(w, z, z, rt) + Q(z, z, z, rt) + Q(w, zz, rt) \phi(t) dt \]
\[ \geq \int_0^t Q(w, z, z, rt) \phi(t) dt \]
Therefore we have \( z = w \) and by Lemma 2.11 \( z \) is unique common fixed point of \( f, g, S \) and \( T \). The uniqueness of the fixed point holds from (ii).

**Theorem 3.2**: Let \( f, T, g \) and \( S \) a self mapping of a complete symmetric Q-fuzzy metric space with continuous t-norm satisfying the following condition

(i) The pair \( \{ f, T \} \) and \( \{ g, S \} \) be owc.

(ii) For all \( x, y \) in \( X, k \) in \((0,1)\), t>0 such that

\[
\phi(t)dt \geq \int_0^1 \phi(t)dt
\]

where \( \phi: \mathbb{R} \to \mathbb{R} \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that \( \int_0^\epsilon \phi(t)dt > 0 \) for each \( \epsilon > 0 \) and \( \phi: [0,1] \to [0,1] \) such that \( \phi(t) > t \) for all \( 0 < t < 1 \), then there exist a unique common fixed point of \( f, g, S \) and \( T \).

**Proof**: The proof follows from Theorem 3.1

**Theorem 3.3**: Let \( f, T, g \) and \( S \) a self mapping of a complete symmetric Q-fuzzy metric space with continuous t-norm satisfying the following condition

(i) The pair \( \{ f, T \} \) and \( \{ g, S \} \) be owc.

(ii) For all \( x, y \) in \( X, k \) in \((0,1)\), t>0 such that

\[
\int_0^1 \phi(t)dt \geq \int_0^1 \phi(t)dt
\]

where \( \phi: \mathbb{R} \to \mathbb{R} \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that \( \int_0^\epsilon \phi(t)dt > 0 \) for each \( \epsilon > 0 \) and \( \phi: [0,1] \to [0,1] \) such that \( \phi(t) > t \) for all \( 0 < t < 1 \), then there exist a unique common fixed point of \( f, g, S \) and \( T \).

**Proof**: Let the pair \( \{ f, T \} \) and \( \{ g, S \} \) be owc, so there are points \( x, y \in X \) such that \( f x = T x \) and \( g y = S y \). We claim that \( f x = g y \). If not by inequality (ii)

\[
\int_0^1 \phi(t)dt \geq \int_0^1 \phi(t)dt
\]

Therefore \( f x = g y \) i.e. \( fx = Tx = gy = Sy \).

Suppose that there is another point \( z \) such that \( f z = T z \) then by (ii) we have \( f z = T z = g y = S y \) so \( f x = f z \) and \( w = f x = T x \) is the unique point of coincidence of \( f \) and \( g \) by Lemma 2.11 \( w \) is the only common fixed point of \( f \) and \( g \). Similarly there is a unique point \( z \in X \) such that \( z = gz = Sz \).

Assume that \( w \neq z \). We have

\[
\int_0^1 \phi(t)dt \geq \int_0^1 \phi(t)dt
\]
\[ z = w \] by Lemma 2.11 and \( z \) is unique common fixed point of \( f, g, S \) and \( T \). The uniqueness of the fixed point holds from (ii).

References


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