New Treatment of Initial Boundary Problems for
Fourth-Order Parabolic Partial Differential Equations
Using Variational Iteration Method

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Abstract

In this paper, a new technique which is applying to treatment of initial boundary problems for variable coefficient fourth-order parabolic partial differential equations by mixed initial and boundary conditions together to obtain a new initial solution at every iteration using variational iteration method (VIM). The structure of a new successive initial solutions can give a more accurate solution.

Keywords: initial boundary problems, fourth-order parabolic partial differential equations, variational iteration method

1. Introduction

Many researchers discussed the initial and boundary value problems. Exact and numerical solutions for non-linear Burger's equation by variational iteration method was applied by J. Biazar and H. Aminikhah [2009]. The Adomian decomposition method discussed for solving higher dimensional initial boundary value problems by A. M. Wazwaz [2000]. Solving higher dimensional initial boundary value problems by
variational iteration decomposition method by M. A. Noor and S. T. Mohyud-Din [2008]. Analytic treatment for variable coefficient fourth-order parabolic partial differential equations discussed by A. M. Wazwaz [2000, 2001]. Weighted algorithm based on the homotopy analysis method is applied to inverse heat conduction problems and discussed by A. Shidfar and A. Molabahrami [2010]. The boundary value problems was applied by Z. Niu and C. Wang [2010] to calculate a one step optimal homotopy analysis method for linear and nonlinear differential equations with boundary conditions only, and homotopy perturbation technique for solving two-point boundary value problems–compared it with other methods was discussed by C. Chun and R. Sakthivel [(2010)]. the solution of fractional heat-like and wave-like equations with variable coefficients and initial boundary conditions using the decomposition method was found by S. Momani [2005] and so as by using variational iteration method was found by R. Yulita Molliq et.al [2009]. Fractional differential equations with initial boundary conditions by modified Riemann–Liouville derivative was solved by G. Wu and E. W. Lee [2010]. It is worth mentioning that the origin of variational iteration method can be traced back by Inokuti et al. [1978].

It is interesting to point out that all these researchers obtained the solutions of initial and boundary value problems by using either initial or boundary conditions only. So we present a reliable framework by applying a new technique for treatment initial and boundary value problems by mixed initial and boundary conditions together to obtain a new initial solution at every iteration for fourth-order parabolic partial differential equations with variable coefficients using variational iteration method. These technique to construct a new successive initial solutions can give a more accurate solution, some examples are given in this paper to illustrate the effectiveness and convenience of this technique.

2. Variational iteration method

In this section, we introduce the basic idea underlying the variational iteration method (VIM) for solving nonlinear equations. Consider the general nonlinear differential equation

$$Lu + Nu = g(x, t),$$  \hspace{1cm} (2.1)

where $L$ is a linear differential operator, $N$ is a nonlinear operator, and $g$ is a given analytical function. The essence of the method is to construct a correction functional of the form [Biazar J. et.al (2010), C. A. Gomez and A. H. Salas (2010)]
New treatment of initial boundary problems

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(t, s)(Lu_n(x, s) + Nu_n(x, s) - g(x, s))ds, \]

where \( \lambda \) is a Lagrange multiplier which can be identified optimally via the variational theory, \( u_n \) is the approximate solution and \( \delta \tilde{u}_n \) denotes the restricted variation, i.e. \( \delta \tilde{u}_n = 0 \). After determining the Lagrange multiplier \( \lambda \) and selecting an appropriate initial function \( u_0 \), the successive approximations \( u_n \) of the solution \( u \) can be readily obtained. Consequently, the solution of Eq.(2.1) is given by \( u = \lim_{n \to \infty} u_n \).

3. New technique by mixed initial boundary conditions for solving fourth-order parabolic partial differential equation using VIM

To convey the basic idea for treatment of initial boundary value problems by variational iteration method to solve the variable coefficient fourth-order parabolic partial differential equation of the form [A. Q. M. Khaliq and E. H. Twizell,1987]

\[ \frac{\partial^2 u}{\partial t^2} + \mu(x) \frac{\partial^4 u}{\partial x^4} = 0, \mu(x) > 0, l_0 < x < l_1, t > 0, \]

the initial conditions associated with (3.1) are of the form

\[ u(x, 0) = f_0(x), \quad l_0 \leq x \leq l_1, \]
\[ \frac{\partial u}{\partial t}(x, 0) = f_1(x), \quad l_0 \leq x \leq l_1, \]

and the boundary conditions are given by

\[ u(l_0, t) = g_0(t), \quad u(l_1, t) = g_1(t), \quad t > 0, \]
\[ \frac{\partial^2 u}{\partial x^2}(l_0, t) = h_0(t), \quad \frac{\partial^2 u}{\partial x^2}(l_1, t) = h_1(t), \quad t > 0, \]

where the functions \( f_0(x), f_1(x), g_0(t), g_1(t), h_0(t) \) and \( h_1(t) \) are continuous functions. The initial solution can be written as \( u_0(x, t) = f_0(x) + tf_1(x) \).

According to the variational iteration method [J. Biazar et.al (2010)], we have a correction functional as follows
\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(t, s) \left( \frac{\partial^2 u_n}{\partial s^2} + \mu(x) \frac{\partial^4 u_n}{\partial x^4} \right) ds, \]

where \( \bar{u}_n \) is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as \( \lambda = s - t \), which yields the following iteration formula

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t (s - t) \left( \frac{\partial^2 u_n}{\partial s^2} + \mu(x) \frac{\partial^4 u_n}{\partial x^4} \right) ds. \quad (3.2) \]

The successive approximations \( u_n \) of the solution \( u \) can be readily obtained but must be satisfying the initial and boundary conditions together. So in this paper we construct a new successive solutions \( u_n^* \) by applying a new technique

\[ u_n^*(x, t) = u_n(x, t) + \left( 1 - \frac{x - l_0}{l_1 - l_0} \right) \left[ g_0(t) - u_n(l_0, t) + \frac{x - l_0}{l_1 - l_0} \left( \frac{1}{6} x(l_1 - l_0) - \frac{1}{6} (2l_1 - l_0)(l_1 - l_0) \right) \right] + \left( 1 - \frac{x - l_0}{l_1 - l_0} \right) \left( \frac{1}{6} x(l_1 - l_0) + \frac{1}{6} (2l_0 - l_1)(l_1 - l_0) \right) \left( h_1(t) - \frac{\partial^2 u_n}{\partial x^2} (l_1, t) \right), \quad (3.3) \]

where \( n = 0, 1, 2, \ldots \) and \( u_0(x, t) = f_0(x) + t f_1(x) \).

It is clear that the new successive initial solutions \( u_n^* \) in Eq. (3.3) satisfying the initial and boundary conditions together as follows

- if \( x = l_0 \) then \( u_n^*(l_0, t) = g_0(t) \), and \( \frac{\partial^2 u_n^*(l_0, t)}{\partial x^2} = h_0(t) \),
- if \( x = l_1 \) then \( u_n^*(l_1, t) = g_1(t) \), and \( \frac{\partial^2 u_n^*(l_1, t)}{\partial x^2} = h_1(t) \),
- if \( t = 0 \) then \( u_n^*(x, 0) = u_n(x, 0) \).

Now, by using Eq. (3.3) we can be rewritten Eq. (3.2) in a new formulation to obtain the correct functional as

\[ u_{n+1}(x, t) = u_n^*(x, t) + \int_0^t (s - t) \left( \frac{\partial^2 u_n^*}{\partial s^2} + \mu(x) \frac{\partial^4 u_n^*}{\partial x^4} \right) ds. \quad (3.4) \]

Such as treatment is a very effective as shown in this paper.
4. Applications the new successive solutions $u^*_n$ for variable coefficient fourth-order parabolic partial differential equations

**Example 1:** Consider the parabolic equation [Khaliq A. Q. M. and Twizell E.H.,1987; Wazwaz A.M.,1995]

$$\frac{\partial^2 u}{\partial t^2} + \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, \quad t > 0,$$

subject to the initial conditions

$$u(x,0) = 0, \quad \frac{1}{2} < x < 1,$$

$$\frac{\partial u}{\partial t} (x,0) = 1 + \frac{x^5}{120}, \quad \frac{1}{2} < x < 1,$$

and the boundary conditions

$$u\left(\frac{1}{2}, t\right) = \left(1 + \frac{(1/2)^5}{120}\right) \sin t, \quad u(1,t) = \frac{121}{120}\sin t, \quad t > 0,$$

$$\frac{\partial^2 u\left(\frac{1}{2}, t\right)}{\partial x^2} = \frac{1}{(1/2)^3} \sin t, \quad \frac{\partial^2 u(1,t)}{\partial x^2} = \frac{1}{6} \sin t, \quad t > 0.$$

By applying a new approximations $u^*_n$ in Eq. (3.3) we obtain

$$u^*_n(x,t) = u_n(x,t) + (2 - 2x) \left[ \left(1 + \frac{(1/2)^5}{120}\right) \sin t - u_n(1/2,t) + (2x - 1) \left(\frac{x}{12} - \frac{1}{6}\right) \left(\frac{1}{(1/2)^3} \sin t - \frac{\partial^2 u_n(1/2,t)}{\partial x^2}\right) \right] + (2x - 1) \left[\frac{121}{120}\sin t - u_n(1,t) + (2 - 2x) \left(\frac{x}{12} - \frac{1}{6}\right) \left(\frac{1}{6} \sin t - \frac{1}{48}\right) \right],$$

where $n = 0, 1, 2, \ldots$. The initial approximation is $u_0(x,t) = \left(1 + \frac{x^5}{120}\right) t$. Now, we begin with a new initial approximation $u^*_0 (when n = 0)$

$$u^*_0(x,t) = \left(1 + \frac{x^5}{120}\right) t + (2 - 2x) \left[ \frac{3841}{3840}\sin t - \frac{3841}{3840} t + (2x - 1) \left(\frac{x}{12} - \frac{1}{6}\right) \left(\frac{1}{48}\right) \sin t - \frac{1}{48}\right] + (2x - 1) \left[\frac{121}{120}\sin t - \frac{121}{120} t - \frac{1}{12} \left(2 - 2x\right) x \left(\frac{1}{6} \sin t - \frac{1}{6}\right) \right].$$

Substituting Eq. (4.1) by Eq. (3.4) we have a correction functional as follows

$$u_{n+1}(x,t) = u^*_n(x,t) + \int_0^t (s-t) \left( \frac{\partial^2 u^*_n}{\partial s^2} + \frac{1}{x} + \frac{x^4}{120} \frac{\partial^4 u^*_n}{\partial x^4} \right) ds.$$

Consequently, the following approximants are obtained
$u_1(x, t) = \left(1 + \frac{x^5}{120}\right) t - \left(1 + \frac{x^5}{120}\right) \frac{t^3}{3!}$,

$u_2(x, t) = \left(1 + \frac{x^5}{120}\right) t - \left(1 + \frac{x^5}{120}\right) \frac{t^3}{3!} + \left(1 + \frac{x^5}{120}\right) \frac{t^5}{5!}$,

and so on. The solution in a series form is

$u(x, t) = \left(1 + \frac{x^5}{120}\right) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$,

and therefore the exact solution is

$u(x, t) = \left(1 + \frac{x^5}{120}\right) \sin t$.

**Example 2:** We next consider the parabolic equation [A. M. Wazwaz, 2001]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{x}{\sin x} - 1\right) \frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < x < 1, t > 0,$$

(4.2)

subject to the initial conditions

$u(x, 0) = x - \sin x, \quad 0 < x < 1,$

$\frac{\partial u}{\partial t} (x, 0) = -(x - \sin x), \quad 0 < x < 1,$

and the boundary conditions

$u(0, t) = 0, \quad u(1, t) = e^{-t}(1 - \sin 1), \quad t > 0,$

$\frac{\partial^2 u(0, t)}{\partial x^2} = 0, \quad \frac{\partial^2 u(1, t)}{\partial x^2} = e^{-t} \sin 1, \quad t > 0.$

By applying a new approximations $u_n^*$ in Eq. (3.3) we obtain

$u_n^*(x, t) = u_n(x, t) + (1 - x) \left[0 - u_n(0, t) + x \left(\frac{x}{6} - \frac{1}{3}\right) \left(0 - \frac{\partial^2 u_n(0, t)}{\partial x^2}\right)\right] + x \left[e^{-t}(1 - \sin 1) - u_n(1, t) + (1 - x) \left(-\frac{x}{6} - \frac{1}{6}\right) \left(e^{-t} \sin 1 - \frac{\partial^2 u_n}{\partial x^2}(1, t)\right)\right],$

where $n = 0, 1, 2, \ldots$. The initial approximation is $u_0(x, t) = (x - \sin x)(1 - t)$. Now, we begin with a new initial approximation $u_0^*$ (when $n = 0$)

$u_0^*(x, t) = (x - \sin x)(1 - t) + x \left[e^{-t}(1 - \sin 1) - (1 - \sin 1)(1 - t) + (1 - x) \left(-\frac{x}{6} - \frac{1}{6}\right) (e^{-t} \sin 1 - (\sin 1)(1 - t))\right].$

Substituting Eq. (4.2) by Eq. (3.4) we have a correction functional as follows

$u_{n+1}(x, t) = u_n^*(x, t) + \int_0^t (s - t) \left(\frac{\partial^2 u_n^*}{\partial s^2} + \left(\frac{x}{\sin x} - 1\right) \frac{\partial^4 u_n^*}{\partial x^4}\right) ds.$
Consequently, the following approximants are obtained

\[ u_1(x, t) = (x - \sin x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}), \]

\[ u_2(x, t) = (x - \sin x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}), \]

and so on. The solution in a series form is

\[ u(x, t) = (x - \sin x) \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right), \]

and therefore the exact solution is

\[ u(x, t) = (x - \sin x) e^{-t}. \]

**Example 3:** We finally consider the nonhomogeneous parabolic equation [A. M. Wazwaz, 2001]

\[ \frac{\partial^2 u}{\partial t^2} + (1 + x) \frac{\partial^4 u}{\partial x^4} = \left(x^4 + x^3 - \frac{6}{7!} x^2\right) \cos t, \quad 0 < x < 1, t > 0, \]  

subject to the initial conditions

\[ u(x, 0) = \frac{6}{7!} x^7, \quad 0 < x < 1, \]

\[ \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < 1, \]

and the boundary conditions

\[ u(0, t) = 0, \quad u(1, t) = \frac{6}{7!} \cos t, \quad t > 0, \]

\[ \frac{\partial^2 u}{\partial x^2}(0, t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1, t) = \frac{1}{20} \cos t, \quad t > 0. \]

By applying a new approximations \( u^*_n \) in Eq. (3.3) we obtain

\[ u^*_n(x, t) = u_n(x, t) + (1 - x) \left[ 0 - u_n(0, t) + x \left( -\frac{x}{6} - \frac{1}{6} \right) \left(0 - \frac{\partial^2 u_n(0, t)}{\partial x^2}\right)\right] + x \left[ \frac{6}{7!} \cos t - \right] \]

\[ u_n(1, t) + (1 - x) \left( -\frac{x}{6} - \frac{1}{6} \right) \left( \frac{1}{20} \cos t - \frac{\partial^2 u_n}{\partial x^2}(1, t)\right) \]

where \( n = 0, 1, 2, \ldots \). The initial approximation is \( u_0(x, t) = \frac{6}{7!} x^7 \). Now, we begin with a new initial approximation \( u^*_0 \) (when \( n = 0 \))

\[ u^*_0(x, t) = \frac{6}{7!} x^7 + x \left[ \frac{6}{7!} \cos t - \frac{6}{7!} + (1 - x) \left( -\frac{x}{6} - \frac{1}{6} \right) \left( \frac{1}{20} \cos t - \frac{1}{20}\right)\right]. \]

Substituting Eq. (4.3) by Eq. (3.4) we have a correction functional as follows
\[ u_{n+1}(x, t) = u_n^*(x, t) \]
\[ + \int_0^t (s - t) \left( \frac{\partial^2 u_n^*}{\partial s^2} + (1 + x) \frac{\partial^4 u_n^*}{\partial x^4} - \left( x^4 + x^3 - \frac{6}{7} x^7 \right) \cos s \right) \, ds. \]

Consequently, we obtain

\[ u_1(x, t) = -\frac{1}{2} t^2(x^3 + x^4) + (x^3 + x^4)(1 - \cos t) + \frac{1}{840} x^7 \cos t, \]
\[ u_2(x, t) = 24(1 + x)(1 - \cos t - \frac{t^2}{2!} + \frac{t^4}{4!}) + \frac{1}{840} x^7 \cos t, \]
\[ u_3(x, t) = \frac{1}{840} x^7 \cos t. \]

Which is the exact solution.

5. Conclusions

In this paper, a very effective to construct a new initial successive solutions \( u_n^* \) by mixed initial and boundary conditions together which explained in formula (3.3) and used it to find successive approximations \( u_n \) of the solution \( u \) in a new correct functional which explained in Eq.(3.4) by applying variational iteration method to solve initial boundary problems for variable coefficient fourth-order parabolic partial differential equations. These technique to construct of a new successive initial solutions can give a more accurate solution. Some examples are given in this paper to illustrate the effectiveness and convenience of a new technique.

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References


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