A Condition for the Equivalence of $I$- and $I^*$-Convergence in 2-Normed Spaces

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Abstract
In this article we define and investigate the concepts of $I$ and $I^*$-convergence for double sequences in 2-normed space and derive several basic properties of these concepts in 2-normed space. Finally, we consider the condition (AP2) of $I$ and $I$-convergence of double sequences in 2-normed spaces and prove some basic properties.

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1 Introduction

The concept of statistical convergence was introduced over nearly the last fifty years (Fast (1951), Schoenberg (1959)), it has become an active area of research in recent years. The idea of $I$-convergence was informally introduced by Kostyrko et al (2001) and also independently by Nuray and Ruckle (2000), as a generalization of statistical convergence. Ideal convergence provides a general framework to study the properties of various types of convergence. Very recently some works on $I$-convergence of double sequences have also been done (see [5,6,21,26]). The notion of linear 2-normed spaces has been investigated by Gähler in 1960’s [11,12] and has been developed extensively in different subjects by others[3,14,23,27]. It seems therefore reasonable to investigate the concepts of $I$ and $I^*$-convergence for the double sequences in 2-normed spaces.

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|\|)$ be a normed space. Let $E$ be subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$. The quotient $d_j(E) = \text{card}(E \cap \{1, \ldots, j\})/j$ is called the $j$th partial density of $K$. Note that $d_j$ is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, \ldots, j\}$ [2,4,7,26].

The limit $d(E) = \lim_{j \to \infty} d_j(E)$ is called the natural density of $E \subseteq \mathbb{N}$ (if exists). Clearly, finite subsets have natural density zero and $d(E^c) = 1 - d(E)$ where $E^c = E - \mathbb{N}$, i.e., the complement of $E$ [2,24].

Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$ is said to be statistically convergent to $l \in X$ if the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - l\| \geq \epsilon\}$ has natural density zero for each $\epsilon > 0$ in other words for each $\epsilon > 0$,

$$\lim_{n} \frac{1}{n} \text{card}(\{k \leq n : |x_k - l| \geq \epsilon\}) = 0$$

and $x=(x_n)_{n \in \mathbb{N}}$ is called to be statistically Cauchy sequence if for each $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$\lim_{n} \frac{1}{n} \text{card}(\{k \leq n : |x_k - x_N| \geq \epsilon\}) = 0$$

The convergence of a double sequence introduce by many manner[4,21,22]. By the convergence of a double sequence we mean the convergence in Pring-
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Shein's sense [22]. A double sequence $x=(x_{jk})_{j,k \in \mathbb{N}}$ is called to be convergent in the Pringsheim's sense if for each $\varepsilon > 0$ there exist a positive integer $N = N(\varepsilon)$ such that for all $j, k \geq N$ implies $|x_{jk} - l| < \varepsilon$. $L$ is called the Pringsheim limit of $x$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set of positive integers and let $A(n, m)$ be the numbers of $(j, k)$ in $A$ such that $j \leq n$ and $k \leq m$. Then the two-dimensional concept of natural density can be defined as follows.

The lower asymptotic density of a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$d_2(A) = \liminf_{n,m} \frac{A(n,m)}{nm}$$

If the sequence $(\frac{A(n,m)}{nm})_{n,m \in \mathbb{N}}$ has a limit in Pringsheim's sense then we say that $A$ has a double natural density and is defined as

$$d_2(A) = \lim_{n,m} \frac{A(n,m)}{nm}$$

Next we recall the following definition, where $Y$ represents an arbitrary set.

**Definition 1.1.** A family $\mathcal{I} \subseteq \mathcal{P}(Y)$ of subsets a nonempty set $Y$ is said to be an ideal in $Y$ if:

i) $\emptyset \in \mathcal{I}$

ii) $A, B \in \mathcal{I}$  implies  $A \cup B \in \mathcal{I}$

iii) $A \in \mathcal{I}, B \subseteq A$  implies  $B \in \mathcal{I}$

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

**Definition 1.2.** Let $Y \neq \emptyset$. An empty family $F$ of subsets of $Y$ is said to be a filter in $Y$ provided:

i) $\emptyset \in F$.

ii) $A, B \in F$  implies  $A \cap B \in F$.

iii) $A \in F, A \subseteq B$  implies  $B \in F$.

If $\mathcal{I}$ is a nontrivial ideal in $Y, Y \neq \emptyset$, then the class

$$F(\mathcal{I}) = \{M \subset Y : (\exists A \in \mathcal{I})(M = Y - A)\}$$

is a filter on $Y$, called the filter associated with $\mathcal{I}$.

**Definition 1.3.** A nontrivial ideal $\mathcal{I}$ in $Y$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in Y$. 
Definition 1.4. A nontrivial ideal $\mathcal{I}$ in $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if 
$\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Let $\mathcal{I}_0 = \{ A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \in \mathbb{N} \times \mathbb{N} - A)\}$.

Then $\mathcal{I}_0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}$ is strongly admissible if and only if $\mathcal{I}_0 \subseteq \mathcal{I}$.\[5\]

Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{ n \in \mathbb{N} : \|x_n - x\| \geq \epsilon \}$ belongs to $\mathcal{I}$ \[1,17,19\].

2 Preliminary Notes

In \[10\], the concept of $I^*$-convergence was also introduced and a detailed study was made to explore its relation with $I$-convergence. We introduce the concepts $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences in metric space $X$ is follow.

Definition 2.1. A double sequence $x = (x_{jk})_{j,k \in \mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}$-convergent to $l \in X$ if for every $\epsilon > 0$ we have $A(\epsilon) \in \mathcal{I}$, where $A(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, l) \geq \epsilon \}$ and we write it as

$$\mathcal{I} - \lim_{m,n} x_{mn} = l$$

Definition 2.2. A double sequence $x = (x_{jk})_{j,k \in \mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}^*$-convergent to $l \in X$ if there exists a set $M \in F(\mathcal{I})$ (i.e. $\mathbb{N} \times \mathbb{N} - M \in \mathcal{I}$) such that

$$\lim_{m,n} x_{mn} = l \quad (m, n) \in M$$

We write it as

$$\mathcal{I}^* - \lim_{m,n} x_{mn} = l$$

The notion of linear 2-normed spaces has been investigated by Gâhler in 1960’s \[11,12\] and has been developed extensively in different subjects by others\[3,14,23\]. Let $X$ be a real linear space of dimension greater than 1, and $\|.,.\|$ be a non-negative real-valued function on $X \times X$ satisfying the following conditions:
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G1) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent vectors.

G2) $\|x, y\| = \|y, x\|$ for all $x, y$ in $X$.

G3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ where $\alpha$ is real.

G4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z$ in $X$.

$\|., .\|$ is called a 2-norm on $X$ and the pair $(X, \|., .\|)$ is called a linear 2-normed space. In addition, for all scalars $\alpha$ and all $x, y, z$ in $X$, we have the following properties:

1) $\|., .\|$ is nonnegative.

2) $\|x, y\| = \|x, y + \alpha x\|$

3) $\|x - y, y - z\| = \|x - y, x - z\|

Some of the basic properties of 2-norm introduce in [24].

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors $x$ and $y$, which may be given clearly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2) \quad y = (y_1, y_2) \quad (2.1)$$

Given a 2-normed space $(X, \|., .\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is said to be convergent to $x$ in $X$ if $\lim_{n \to \infty} \|x_n - x, z\| = 0$ for every $z \in X$. This can be written by the formula:

$$(\forall z \in Y)(\forall \epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad \|x_n - x, z\| < \epsilon$$

We write it as

$$x_n \xrightarrow{\|., .\|_X} x$$

**Definition 2.3.** A sequence $(x_n)_{n \in \mathbb{N}}$ in a 2-normed space $(X, \|., .\|)$ is a Cauchy sequence if $\lim_{n,m} \|x_n - x_m, z\| = 0$ for every $z \in X$.

Recall that $(X, \|., .\|)$ is a 2-Banach space, if every Cauchy sequence in $X$ is convergence to some $x \in X$.

**Definition 2.4.** Let $(X, \|., .\|)$ be 2-normed space and $x \in X$. We say that $x$ is an accumulation point of $X$ if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of distinct elements of $X$ such that $x_k \neq x$ (for any $k$) and $x_n \xrightarrow{\|., .\|_X} x$.
Lemma 2.5. [15] Let \( v = \{ v_1, \ldots, v_k \} \) be a basis of \( X \). A sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) is convergent to \( x \) in \( X \) if and only if \( \lim_{n \to \infty} \| x_n - x, v_i \| = 0 \) for every \( i = 1, \ldots, k \). We can define the norm \( \| \|_\infty \) on \( X \) by

\[
\| x \|_\infty := \max\{ \| x, v_i \| : i = 1, \ldots, d = k \}
\]

Associated to the derived norm \( \| \|_\infty \), we can define the (open) balls \( B_{v_1,v_2,\ldots,v_n}(x,r) = B_v(x,r) \) centered at \( x \) having radius \( r \) by

\[
B_v(x,r) := \{ y : \| x - y \|_\infty < r \}
\]

Lemma 2.6. [15] A sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) is convergent to \( x \) in \( X \) if and only if

\[
\lim_{n \to \infty} \| x_n - x \|_\infty = 0
\]

Example 2.7. Let \( X = \mathbb{R}^2 \) be equipped with the 2-norm \( \| x, y \| := \) the area of the parallelogram spanned by the vectors \( x \) and \( y \), which may be given explicitly by the formula

\[
\| x, y \| = |x_1y_2 - x_2y_1| , \quad x = (x_1, x_2) \quad y = (y_1, y_2)
\]

Take the standard basis \( \{ i, j \} \) for \( \mathbb{R}^2 \).

Then, \( \| x, i \| = |x_2| \) and \( \| x, j \| = |x_1| \), and so the derived norm \( \| \|_\infty \) with respect to \( \{ i, j \} \) is

\[
\| x \|_\infty = \max\{ |x_1|, |x_1| \}, \quad x = (x_1, x_2)
\]

Thus, here the derived norm \( \| \|_\infty \) is exactly the same as the uniform norm on \( \mathbb{R}^2 \). Since the derived norm is norm, it is equivalent to Euclidean norm on \( \mathbb{R}^2 \).

3 Main Results

Now, we introduce the \( I_2 \) and \( I_2^* \)-convergence for double sequences in 2-normed spaces and so we extend this concepts to \( I_2 \)-limit points and \( I_2 \)-cluster points in this spaces. We inspire the follow definition of Pringsheim sense[22].
Definition 3.1. Let \( x = (x_{jk})_{j,k \in \mathbb{N}} \) be a double sequence in 2-normed space \((X, \|\cdot\|)\). A double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) is said to be convergent to \( l \in X \) if
\[
(\forall z \in X)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall j, k \geq N) \quad \|x_{jk} - l, z\| < \varepsilon
\]
We write it as
\[
x_{jk} \xrightarrow{\|\cdot\|_X} l
\]
A double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) is said to be bounded if for each nonzero \( z \in X \) and for all \( j, k \in \mathbb{N} \) there exists \( M > 0 \) such that \( \|x_{jk}, z\| < M \).
Note that a convergent double sequence need not be bounded.

Now we define the \( I_2 \) and \( I_2^* \)-convergence for double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) as follows:

Definition 3.2. A double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) in 2-normed space \((X, \|\cdot\|)\) is said to be \( I_2 \)-convergence to \( l \in X \), if for all \( \varepsilon > 0 \) and nonzero \( z \in X \), the set
\[
A(\varepsilon) = \{(j, k) : \|x_{jk} - l, z\| \geq \varepsilon\} \in I_2
\]
In this case we write it as
\[
I_2 - \lim_{j,k} x_{jk} = l
\]

Remark 3.3. Put \( I_d = \{A \subset \mathbb{N} \times \mathbb{N} : d_2(A) = 0\} \). Then \( I_d \) is an admissible ideal in \( \mathbb{N} \times \mathbb{N} \) and \( I_2^* \)-convergence becomes statistical convergence[24].

Remark 3.4. If \( x = (x_{jk})_{j,k \in \mathbb{N}} \) is \( I_2 \)-convergent, then \( (x_{jk})_{j,k \in \mathbb{N}} \) need not be convergent. Also it is not necessarily bounded. This actuality can be seen from the next example.

Definition 3.5. A double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) in 2-normed space \((X, \|\cdot\|)\) is said to be \( I_2^* \)-convergence to \( l \in X \), if there exists a set \( M \in F(I) \) (i.e. \( \mathbb{N} \times \mathbb{N} \setminus M \in I \)) such that \( \lim_{m,n} x_{mn} = l \) \((m,n) \in M \) and we write it
\[
I_2^* - \lim_{j,k} x_{jk} = l
\]

Example 3.6. Let \((X, \|\cdot\|)\) be 2-normed space introduced in Example 2.3 and the \( x = (x_{jk})_{j,k \in \mathbb{N}} \) be defined as

\[
(1,1) \text{ otherwise}
\]
and let \( l = (1, 1) \).

Then for every \( \varepsilon > 0 \) and \( z \in X \)
\[
\{(j, k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \subseteq \{1, 4, 9, 16, \ldots, j^2, \ldots\} \times \{1, 4, 9, 16, \ldots, k^2, \ldots\}.
\]

Hence
\[
\text{the cardinality of the set } \{(j, k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \leq \sqrt{j} \sqrt{k} \text{ for each } \varepsilon > 0.
\]

This implies \( d_2(\{(j, k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\}) = 0 \) for each \( \varepsilon > 0 \)
and \( z \in X \). We have
\[
I_{d_2} - \lim_{j,k} x_{jk} = l
\]

But \( x = (x_{jk})_{j,k \in \mathbb{N}} \) is neither convergent to \( l \) nor bounded.

**Remark 3.7.** The following corollary can be verifies that if \( x = (x_{jk})_{j,k \in \mathbb{N}} \) be \( I_2 \)-convergent to \( l \in X \), then \( l \) is determined uniquely.

**Corollary 3.8.** let \( x = (x_{jk})_{j,k \in \mathbb{N}} \) is a convergent double sequence in 2-normed space \((X, \|\cdot, \cdot\|)\) and \( l_1, l_2 \in X \). If \( I_2 - \lim_{j,k} \|x_{jk} - l_1, z\| = 0 \)
and \( I_2 - \lim_{j,k} \|x_{jk} - l_2, z\| = 0 \) then \( l_1 = l_2 \).

**proof:** Let \( l_1 \neq l_2 \), hence there exists \( z \in X \) such that \( 0 \neq l_1 - l_2 \) and \( z \) are linearly independent. Put
\[
\|l_1 - l_2, z\| = 2\varepsilon, \text{ with } \varepsilon > 0
\]

Now
\[
2\varepsilon = \|l_1 - x_{jk} + x_{jk} - l_2, z\| \leq \|x_{jk} - l_1, z\| + \|x_{jk} - l_2, z\|
\]

Therefore
\[
\{(j, k) : \|x_{jk} - l_2, z\| < \varepsilon\} \subseteq \{(j, k) : \|x_{jk} - l_1, z\| \geq \varepsilon\} \in I
\]

Hence \( \{j, k) : \|x_{jk} - l_2, z\| < \varepsilon\} \in I \) that is contradict with nontrivial \( I \).

**Corollary 3.9.** If \( (x_{jk})_{j,k \in \mathbb{N}}, (y_{jk})_{j,k \in \mathbb{N}} \) be double sequences in 2-normed space \((X, \|\cdot, \cdot\|)\) and \( I_2 - \lim_{j,k} x_{jk} = a, I_2 - \lim_{j,k} y_{jk} = b \) then

(i) \( I_2 - \lim_{j,k} x_{jk} + y_{jk} = a + b \)
\( (ii) I_2 \lim_{j,k} \alpha x_{jk} = \alpha a \), where \( \alpha \in \mathbb{R} \)

**proof(i):** Let \( \varepsilon > 0 \). For each nonzero \( z \in X \) we have

\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : \| (x_{mn} + y_{mn}) - (a + b) \| z \geq \varepsilon \} \subseteq \bigcup \{(m, n) \in \mathbb{N} \times \mathbb{N} : \| x_{mn} - a \| z \geq \frac{\varepsilon}{2} \}
\]

Hence \( \{(m, n) \in \mathbb{N} \times \mathbb{N} : \| (x_{mn} + y_{mn}) - (a + b) \| z \geq \varepsilon \} \in I \) and the statements is follows.

\( (ii) \) The statement is an easy consequence of (i)

**Condition (AP2) In 2-Normed Spaces**

In [5], it was proved that \( I \) and \( I^* \)-convergence of double sequence are equivalent if and only if \( I \subseteq 2^{\mathbb{N} \times \mathbb{N}} \) satisfies the condition (AP2). It seems therefore reasonable to consider the condition (AP2) of \( I_2 \) and \( I_2^* \)-convergence of double sequences in 2-normed spaces.

**Definition 3.10.** An admissible ideal \( I \subseteq 2^{\mathbb{N} \times \mathbb{N}} \) satisfies the condition (AP2) if for every countable family of disjoint sets \( \{A_1, A_2, \ldots\} \) belonging to \( I \), there exists a countable family of sets \( \{B_1, B_2, \ldots\} \) such that \( A_j \Delta B_j \in I \), i.e.,

\( A_j \Delta B_j \) is included in the finite union of rows and columns \( \mathbb{N} \times \mathbb{N} \) for each \( j \in \mathbb{N} \) and \( B = \bigcup_{j=1}^{\infty} B_j \in I \) (hence \( B_j \in I \) for each \( j \in \mathbb{N} \)).

**Theorem 4.3:** If \( (X, \|\cdot\|) \) be a finite dimensional 2-normed space and \( I \subseteq 2^{\mathbb{N} \times \mathbb{N}} \) be an admissible ideal that satisfied the condition (AP2), then for an arbitrary double sequence \( (x_{mn})_{m,n \in \mathbb{N}} \) of elements of \( X \),

\[ I_2 - \lim_{m,n} x_{mn} = a \implies I_2^* - \lim_{m,n} x_{mn} = a. \]

**proof:** If \( I_2 - \lim_{m,n} x_{mn} = a \) then for any \( \varepsilon > 0 \) and nonzero \( z \in X \),

\[ A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \| x_{mn} - a \| \geq \varepsilon \} \in I \]

Put \( A_1 = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \| x_{mn} - a \| \geq 1 \} \) and for \( k \geq 2 \)

\[ A_k = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k} \leq \| x_{mn} - a \| < \frac{1}{k-1} \} \] for each nonzero \( z \in X \).

Clearly \( A_i \cap A_j = \emptyset \) for \( i \neq j \) and \( A_i \in I \) for each \( i \in \mathbb{N} \). Hence there exists a countable family of sets \( \{B_1, B_2, \ldots\} \) such that \( A_j \Delta B_j \) is included in finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \) for each \( j \) and \( B = \bigcup_{j=1}^{\infty} B_j \in I \). Put \( M = \mathbb{N} \times \mathbb{N} - B \) and prove \( \lim_{m,n} x_{mn} = a \) for \( (m, n) \in M \).

Let \( \delta > 0 \). Choose \( k \in \mathbb{N} \) such that \( \frac{1}{k} < \delta \). Then
{(m, n) : \|x_{mn} - a\|_{\infty} \geq \delta} \subseteq \bigcup_{j=1}^{n} A_j.

Since \(A_j \Delta B_j, j = 1, 2, ..., k\) are included in finite union of rows and columns, there exists \(n_0 \in \mathbb{N}\) such that
\[
\bigcup_{j=1}^{k} B_j \cap \{(m, n) : m \geq n_0, n \geq n_0\}
\]

If \(m, n \geq n_0\) and \((m, n) \notin B\) then
\[(m, n) \notin \bigcup_{j=1}^{k} B_j \text{ and so } (m, n) \notin \bigcup_{j=1}^{k} A_j.\]

This implies that
\[
\|x_{mn} - a, z\| < \frac{1}{k} < \delta \text{ for each } z \in X.
\]

Hence \(\lim_{m,n} x_{mn} = a\) for \((m, n) \in M\).

**Theorem 4.4:** Let \((X, \|., .\|)\) be 2-normed space. If \(X\) has at least one accumulation point for any arbitrary double sequence \((x_{mn})_{m,n \in \mathbb{N}} \subseteq X\) and for each \(a \in X\), \(I_2 - \lim_{m,n} x_{mn} = a\) implies \(I_2^* - \lim_{m,n} x_{mn} = a\), then \(I\) has the property (AP2).

**Proof:** Let \(a \in X\) is an accumulation point of \(X\). Let \((b_k)_{k \in \mathbb{N}}\) be a sequence of distinct elements of \(X\) such that \(b_k \neq a\) for any \(k\),
\[
b_k \xrightarrow{\|., .\|_X} a
\]

Put \(\varepsilon_k^* = \|b_k - a, z\|\) for \(k \in \mathbb{N}\). Let \((A_j)_{j \in \mathbb{N}}\) be a disjoint family of nonempty sets from \(I\). Define a sequence \((x_{mn})_{m,n \in \mathbb{N}}\) for all \(z \in X\)

\[
a \text{ if } (m, n) \notin A_j
\]

Let \(\delta > 0, z_0 \in X\). Choose \(k \in \mathbb{N}\) such that \(\varepsilon_k^{(z_0)} < \delta\). Then

...
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\[
A^{\omega}(\delta) = \{(m,n): \|x_{mn} - a, z_0\| \geq \delta\} \subseteq A_1 \cup A_2 \cup ... \cup A_k
\]

Hence \( A^{\omega}(\delta) \in I \) and so \( I_2 - \lim_{m,n} x_{mn} = a \).

By definition(...) \( I^*_2 - \lim_{m,n} x_{mn} = a \) and so there exist a set \( B \in I \) such that \( M = \mathbb{N} \times \mathbb{N} - B \in F(I) \) and

\[
x_{mn} \xrightarrow{\|\cdot\|} a \quad (m,n) \in M \tag{3.1}
\]

Put \( B_j = A_j \cap B \) for \( j \in \mathbb{N} \). Then \( B_j \in I \) for each \( j \in \mathbb{N} \).

Moreover \( \bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subseteq B \) and so \( \bigcup_{j=1}^{\infty} B_j \in I \). Fix \( j \in \mathbb{N} \). If \( A_j \cap M \) is not included in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \), then \( M \) must contain an infinite sequence of element \((m_k,n_k)\) and \( \lim x_{m_k,n_k} = b_j \neq a \) for all \( k \in \mathbb{N} \) which it contradicts with (4.1). Hence \( A_j \cap M \) must be contained in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \).

Hence \( A_j \Delta B_j = A_j - B_j = A - B = A_j \cap M \) is also included in the finite union of rows and columns. This proves that the ideal \( I \) has the property (AP2)

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