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Asymptotic Behavior of Green's Function for Boundary Condition Functions for a Second Order Boundary Value Problem

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Abstract

The Green's function of boundary condition functions for a second order boundary value problem is shown to be asymptotically equivalent to the Green's function of the corresponding Fourier problem.

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1 Introduction

Although research on boundary condition functions has not been very prolific lately, E. C. Titchmarsh [8, 9] did some exhaustive work in this field. W. N. Everitt [1, 2, 3] and D. N Offei [4, 5] have also looked at some aspects of boundary condition functions. In [6, 7], the boundary condition functions and the Wronskian of the boundary condition functions for the second order boundary value problem

$$L\phi \equiv -\phi^{(2)}(x) + p(x)\phi(x) = \lambda\phi(x); \quad (a \leq x \leq b) \tag{1}$$

$$U_r\phi \equiv \sum_{s=1}^{2}[\alpha_{rs}\phi^{(s-1)}(a) + \beta_{rs}\phi^{(s-1)}(b)] = 0 \;\; (1 \le r \le 2) \tag{2}$$

where the function p, the boundary constants α_{rs}, β_{rs} and the parameter λ are complex with separated boundary conditions or otherwise were shown to be asymptotically equivalent for large values of $|\lambda|$ to the boundary condition functions and the Wronskian of the boundary condition functions of the corresponding Fourier problem

$$L\phi \equiv -\phi^{(2)}(x) = \lambda\phi(x) \tag{3}$$

$$U_r\phi \equiv \sum_{s=1}^{2}[\alpha_{rs}\phi^{(s-1)}(a) + \beta_{rs}\phi^{(s-1)}(b)] = 0 \quad (1 \le r \le 2) \tag{4}$$

In this paper, we show that the Green's function of the boundary condition functions of equations (1)-(2) is asymptotically equivalent to the Green's function of the boundary condition functions of the corresponding Fourier problem (3)-(4).

Again in [6], we showed that if $\psi_r(a|x,\lambda)$and $\chi_r(b|x,\lambda)$, $(1 \le r \le 2)$ are boundary condition functions for (1)-(2) and $\psi_{Fr}(a|x,\lambda)$ and $\chi_{Fr}(b|x,\lambda)$,$(1 \le r \le 2)$ are boundary condition functions for (3)-(4) then

1. $\psi_r(a|x,\lambda) \sim \psi_{Fr}(a|x,\lambda)$
2. $\chi_r(a|x,\lambda) \sim \chi_{Fr}(a|x,\lambda)$

as $|\lambda| \to \infty$, $(1 \le r \le 2)$. In [7] we also showed that if,

$$\eta_r(x,\lambda) = \psi_r(a|x,\lambda) + \chi_r(a|x,\lambda)$$

$$\eta_{Fr}(x,\lambda) = \psi_{Fr}(a|x,\lambda) + \chi_{Fr}(a|x,\lambda)$$

and

$$W(\lambda) = W(\eta_1(x,\lambda), \eta_2(x,\lambda))(x)$$

$$W_F(\lambda) = W(\eta_{F1}(x,\lambda), \eta_{F2}(x,\lambda))(x)$$

then the zeros of $W(\lambda)$ and $W_F(\lambda)$ are the eigenvalues of (1) and (3) respectively. Moreover, we showed that

$$W(\lambda) \sim W_F(\lambda)$$

for suitably large values of $|\lambda|$, that is avoiding the eigenvalues where $W_F(\lambda)$ has zeros.

2 Preliminaries

In this section we shall give some notation and properties of the linear differential operator L defined by

$$L\phi = P_0(x)\phi^{(2)}(x) + P_1(x)\phi'(x) + P_2(x)\phi(x) \tag{5}$$

For a suitable pair of functions $\phi_1(x)$ and $\phi_2(x)$, the symbol $\Phi(x)$ denotes the 2×2 matrix $\left[\phi_r^{(s-1)}(x)\right]$ $(1 \leq r, s \leq 2)$ so that

$$\Phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix}$$

and

$$W(\phi_1, \phi_2)(x) = det(\Phi)(x)$$

Closely associated with L is another differential expression L^+ called the Lagrange adjoint of L and given by

$$L^+\psi = \overline{P_0}(x)\psi^{(2)}(x) + \left[2\overline{P_0}(x) - \overline{P_1}(x)\right]\psi'(x) + \overline{P_0}^{(2)}(x) - \overline{P_1}'(x) + \overline{P_2}(x)\psi(x) \tag{6}$$

For a suitable pair of functions f and g

$$\int_a^b \left(\overline{g}Lf - f\overline{L}^+\right) dx = [fg](b) - [fg](a). \tag{7}$$

Here $[fg](x)$ is the bilinear form in (f, f') and $(\overline{g}, \overline{g}')$ given by

$$\begin{aligned} [fg] &= \sum_{j=1}^{2}\sum_{k=1}^{2} B_{jk}(x)\overline{g}^{(j-1)}(x)f^{(k-1)}(x) && (8) \\ &= \hat{g}^*(x)B(x)\hat{f}(x) && (9) \end{aligned}$$

where $\hat{f}(x)$ denotes the column vector with components $f(x)$, $f'(x)$ and $\hat{g}^*(x)$ denotes the row vector with components $\overline{g}(x), \overline{g}'(x)$. The matrix $B(x)$ is given by

$$B(x) = \begin{bmatrix} P_1(x) - P_0(x) & P_0(x) \\ -P_0(x) & 0 \end{bmatrix} \tag{10}$$

The notation A^* is used to represent the conjugate transpose of the matrix A. If there exists $k > 0$ such that $|f(x)| \leq k\phi(x)$ for some $x \geq x_0$, then we write

$$f = O(\phi) \ \text{ as } \ x \to \infty \tag{11}$$

Also if $g = O(\psi)$ as $x \to \infty$, then

$$\left.\begin{array}{rcl} f+g &=& O(\phi+\psi) \\ f.g &=& O(\phi.\psi) \\ k.g &=& O(\phi) \end{array}\right\} \tag{12}$$

where k is a constant. Furthermore, $f \sim l\phi$ as $x \to \infty$ where $l \neq 0$ means that $\frac{f}{\phi} \to l$ as $x \to \infty$.

If $\phi(x,\lambda)$ is the solution of $L\phi = \lambda\phi$ and $\psi(x,\lambda)$ is the solution of $L^{+}\psi = \overline{\lambda}\psi$ then

$$[\phi\psi](x_2) - [\phi\psi](x_1) = \int_{x_1}^{x_2} \{\psi L\phi - \phi\overline{L}^{+}\}dx = 0 \ \ a \leq x_1 \leq x_2 \leq b. \tag{13}$$

Thus $[\phi\psi]$ is independent of $x \in [a,b]$. If $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are solutions of $L\phi = \lambda\phi$ and if $x_1, x_2 \in [a,b]$ then $W(\phi_1,\phi_2)(x_1) = W(\phi_1,\phi_2)(x_2)$ so $W(\phi_1,\phi_2)(x)$ is independent of $x \in [a,b]$. For the special case where $P_0(x) = -1$ and $P_2(x) = 0$

$$[fg](x) = W(f,\overline{g})(x). \tag{14}$$

3 Construction of Green's Function in terms of Boundary Condition Functions

We consider the construction of the Green's function for the boundary value problem (1)-(2). Let $\phi_r(a|x,\lambda), \theta_r(b|x,\lambda)(1 \leq r \leq 2)$ be boundary condition functions for the adjoint of (1)-(2), then $\phi_1(x,\lambda), \theta_1(x,\lambda), \phi_2(x,\lambda)$ and $\theta_2(b|x,\lambda)$ are solutions of (1). Define $G(x,t,\lambda)$ by

$$\begin{aligned} G(x,t,\lambda) &\equiv \begin{cases} G_1(x,\lambda) & (a \leq x \leq t) \\ G_2(x,\lambda) & (t \leq x \leq b) \end{cases} \\ &= \begin{cases} a_1(t,\lambda)\phi_1(a|x,\lambda) + a_2(t,\lambda)\phi_2(a|x,\lambda) & (a \leq x \leq t) \\ b_1(t,\lambda)\theta_1(b|x,\lambda) + b_2(t,\lambda)\theta_2(a|x,\lambda) & (t \leq x \leq b) \end{cases} \end{aligned} \tag{15}$$

where the coefficients a_s and $b_s, (1 \leq s \leq 2)$ are to be determined. As a function of x, $G_1(x,t,\lambda)$ satisfies (1) in the interval $(a \leq x \leq t)$ and $G_2(x,t,\lambda)$ satisfies (1) in the interval $(t \leq x \leq b)$. In [6], we showed that the Wronskian of the boundary condition functions satisfied (2), simialrly

$$U_r(G) \equiv W(G_1, \overline{\psi}_r(a)) + W(G_2, \overline{\chi}_r(b)) \tag{16}$$

$$= \sum_{s=1}^{2} \{a_s(t,\lambda) W(\phi_s, \overline{\psi}_r(a)) + b_s(t,\lambda) W(\theta_s, \overline{\chi}_r(b)) \tag{17}$$

$$= 0 \ \ (1 \leq r \leq 2). \tag{18}$$

Thus, the Green's function of the boundary condition functions satisfies (2)

In [6] we showed that if ψ_r and $\chi_r, (1 \leq r \leq 2)$ are boundary condition functions for (1)-(2) then

$$W(\phi_1, \overline{\psi}_r) + W(\phi_1, \overline{\chi}_r) = 0 \tag{19}$$

So that from (18), (19) and the fact that the Wronskian is independent of $x \in [a,b]$ we have

$$a_s(t,\lambda) = -b_s(t,\lambda) \tag{20}$$

Using the property of Green's function that $G(x,t,\lambda)$ be continuous at $x = t$ we have

$$a_1(t,\lambda)\phi_1(t,\lambda) + a_2\phi_2(t,\lambda) - b_1(t,\lambda)\theta_1(t,\lambda) - b_2(t,\lambda)\theta_2(t,\lambda) = 0 \tag{21}$$

As a function of x, $\frac{\partial G}{\partial x}$ is discontinous at $x = t$ with a jump of -1, that is

$$b_1(t,\lambda){\theta_1}'(t,\lambda) + b_2(t,\lambda){\theta_2}'(t,\lambda) - a_1(t,\lambda){\phi_1}'(t,\lambda) + a_2{\phi_2}'(t,\lambda) = -1 \tag{22}$$

From (20), (21) and (22), we have

$$a_1(t,\lambda)[\phi_1(t,\lambda) + \theta_1(t,\lambda)] + a_2[\phi_2(t,\lambda) + \theta_2(t,\lambda)] = 0 \tag{23}$$

and

$$a_1(t,\lambda)[{\phi_1}'(t,\lambda) + {\theta_1}'(t,\lambda)] + a_2[{\phi_2}'(t,\lambda) + {\theta_2}'(t,\lambda)] = 1 \tag{24}$$

Equations (23) and (24) have solutions if $W(\zeta_1, \zeta_2)(x) = w(\lambda) \neq 0$ where $\zeta_r = \phi_r(a|x,\lambda) + \theta_r(b|x,\lambda)(1 \leq r \leq 2)$ that is avoiding the eigenvalues of the adjoint of (1)-(2) where $w(\lambda) = 0$. Solving for $a_1(t,\lambda), a_2(t,\lambda), b_1(t,\lambda)$ and $b_2(t,\lambda)$ from (23) and (24), we obtain

$$\left.\begin{array}{ll} a_1(t,\lambda) = \frac{-\zeta_2}{w(\lambda)} & a_2(t,\lambda) = \frac{\zeta_1}{w(\lambda)} \\ b_1(t,\lambda) = \frac{\zeta_2}{w(\lambda)} & b_2(t,\lambda) = \frac{-\zeta_1}{w(\lambda)} \end{array}\right\} \tag{25}$$

so that

$$G(x,t,\lambda) = \begin{cases} \frac{-\zeta_2(t,\lambda)\phi_1(a|x,\lambda)+\zeta_1(t,\lambda)\phi_2(a|x,\lambda)}{w(\lambda)} (a \le x \le t) \\ \frac{\zeta_2(t,\lambda)\theta_1(b|x,\lambda)-\zeta_1(t,\lambda)\theta_2(b|x,\lambda)}{w(\lambda)} (t \le x \le b) \end{cases}. \tag{26}$$

Similarly, if $G_F(x,t,\lambda)$ is the Green's function for the adjoint of the corresponding Fourier Problem (3)-(4), then $\phi_{F1}(x,\lambda), \phi_{F2}(x,\lambda), \theta_{F1}(x,\lambda)$ and $\theta_{F2}(x,\lambda)$ are solutions of (3) and

$$\begin{aligned} G(x,t,\lambda) &\equiv \begin{cases} G_{F1}(x,t,\lambda) \\ G_{F2}(x,t,\lambda) \end{cases} \\ &= \begin{cases} \frac{-\zeta_{F2}(t,\lambda)\phi_{F1}(a|x,\lambda)+\zeta_{F1}(t,\lambda)\phi_{F2}(a|x,\lambda)}{w_F(\lambda)} \ (a \le x \le t) \\ \frac{\zeta_{F2}(t,\lambda)\theta_{F1}(b|x,\lambda)-\zeta_{F1}(t,\lambda)\theta_{F2}(b|x,\lambda)}{w_F(\lambda)} \ (t \le x \le b) \end{cases} \end{aligned} \tag{27}$$

where $w_F(\lambda) = W(\zeta_{F1}, \zeta_{F2})(x)$, $\zeta_{Fr} = \phi_{Fr} + \theta_{Fr}(1 \le r \le 2)$ and $w_F(\lambda) \ne 0$.

4 Asymptotic Behavior of Green's Function of Boundary Condition Functions

We now show that if $G(x,t,\lambda)$ is the Green's function in terms of the boundary condition functions for (1)-(2)and$G_F(x,t,\lambda)$ is the Green's function in terms of the boundary condition functions for (3)-(4) then

$$G(x,t,\lambda \sim G_F(x,t,\lambda)$$

as $|\lambda| \to \infty$. If $s^2 = \lambda$ then s is of the form $\pm(\mu + i\rho)$. Let $s = (\mu + i\rho)$ be the solution of $s^2 = \lambda$ such that $\mu \ge 0$.

Theorem 4.1

$$G_{F1}(x,t,\lambda) = 0(|s|^{-1}[e^{|\rho|(x+t-b-a)} + e^{|\rho|(x-t)}])$$

$$G_{F2}(x,t,\lambda) = 0(|s|^{-1}[e^{|\rho|(b+a-x-t)} + e^{|\rho|(t-x)}])$$

as $|\lambda| \to \infty$*, that is, avoiding the eigenvalues where* $w_F(\lambda) = 0$.

Proof. From equation (26)

$$G_{F1}(x,t,\lambda)=\frac{-\zeta_{F2}(t,\lambda)\phi_{F1}(a|x,\lambda)+\zeta_{F1}(t,\lambda)\phi_{F2}(a|x,\lambda)}{w_F(\lambda)}(a\leq x\leq t) \quad (28)$$

where $\zeta_{Fr}=\phi_{Fr}(a|t,\lambda)+\theta_{Fr}(b|t,\lambda)(1\leq r\leq 2)$. In [6], we showed that

$$\left.\begin{array}{rcl}\phi_{Fr}(a|x,\lambda) & = & 0(e^{|\rho|(x-a)}) \\ \phi_{Fr}(a|t,\lambda) & = & 0(e^{|\rho|(t-a)}) \\ \theta_{Fr}(b|t,\lambda) & = & 0(e^{|\rho|(b-t)})\end{array}\right\} \quad (29)$$

as $|\lambda|\to\infty,(1\leq r\leq 2)$. From (12) and (29) we have

$$\zeta_{Fr}(t,\lambda)=0(e^{|\rho|(t-a)}+e^{|\rho|(b-t)}). \quad (30)$$

as $|\lambda|\to\infty,(1\leq r\leq 2)$. From (12), (29) and (30)

$$\begin{aligned}\zeta_{Fr}(t|b,\lambda)\phi_{Fr}(a|x,\lambda) &= 0(e^{|\rho|(x-a)}[e^{|\rho|(t-a)}+e^{|\rho|(b-t)}]) && (31)\\ &= 0(e^{|\rho|(x+t-2a)}+e^{|\rho|(x+b-t-a)}) && (32)\end{aligned}$$

as $|\lambda|\to\infty,(1\leq r,s\leq 2)$ so that

$$\begin{aligned}\zeta_{F2}(t,\lambda)\phi_{F1}(a|x,\lambda) &= 0(e^{|\rho|(x+t-2a)}+e^{|\rho|(x+b-t-a)}) && (33)\\ \zeta_{F1}(t,\lambda)\phi_{F2}(a|x,\lambda) &= 0(e^{|\rho|(x+t-2a)}+e^{|\rho|(x+b-t-a)}). && (34)\end{aligned}$$

as $|\lambda|\to\infty$. In [7], we showed that

$$w_F(\lambda)=0(|s|e^{|\rho|(b-a)}). \quad (35)$$

as $|\lambda|\to\infty$. Hence,

$$\begin{aligned}G_{F1} &= 0(\frac{e^{|\rho|(x+t-2a)}+e^{|\rho|(x+b-t-a)}}{|s|e^{|\rho|(b-a)}}) && (36)\\ &= 0(|s|^{-1}e^{|\rho|(x+t-2a-b+a)}+e^{|\rho|(x+b-t-a-b+a)}) && (37)\\ &= 0(|s|^{-1}e^{|\rho|(x+t-b-a)}+e^{|\rho|(x-t)}) && (38)\end{aligned}$$

for suitably large values of $|\lambda|$, that is, avoiding the eigenvalues where $w_F(\lambda)$ has zeros. Similarly

$$G_{F2}(x,t\lambda)=0(|s|^{-1}e^{|\rho|(b+a-x-t)}+e^{|\rho|(t-x)}). \quad (39)$$

for suitably large values of $|\lambda|$.

Theorem 4.2 *$G(x,t,\lambda) \sim G_F(x,t\lambda)$ for suitably large values of $|\lambda|$ that is, $W_F(\lambda) \neq 0$ and $W(\lambda) \neq 0$.*

Proof. From (29),

$$G_1(x,t,\lambda) = \frac{-\zeta_2(t,\lambda)\phi_1(a|x,\lambda) + \zeta_1(t,\lambda)\phi_2(a|x,\lambda)}{w(\lambda)} \quad (a \leq x \leq t) \tag{40}$$

where $\zeta_r = \phi_r(a|t,\lambda) + \theta_r(b|t,\lambda) \ \ (1 \leq r \leq 2)$. In [6], we showed that

$$\left.\begin{array}{rcl} \phi_{Fr}(a|x,\lambda) &=& 0(e^{|\rho|(t-a)} + e^{|\rho|(b-t)}) \\ \phi_r(a|x,\lambda) &=& \phi_{Fr}(a|x,\lambda) + 0(|s|^{-1}e^{|\rho|(x-a)}) \\ \phi_r(a|t,\lambda) &=& \phi_{Fr}(a|t,\lambda) + 0(|s|^{-1}e^{|\rho|(t-a)}) \\ \theta_r(b|t,\lambda) &=& \theta_{Fr}(b|t,\lambda) + 0(|s|^{-1}e^{|\rho|(b-t)}) \end{array}\right\} \tag{41}$$

as $|\lambda| \to \infty, (1 \leq r \leq 2)$. From (12) and (41) we have

$$\begin{aligned} \zeta_r(t,\lambda) &= \phi_{Fr}(a|t,\lambda) + \theta_{Fr}(b|t,\lambda) + 0(|s|^{-1}[e^{|\rho|(t-a)} + e^{|\rho|(b-t)}]) && (42) \\ &= \zeta_{Fr}(t,\lambda) + 0(|s|^{-1}[e^{|\rho|(t-a)} + e^{|\rho|(b-t)}]) && (43) \end{aligned}$$

as $|\lambda| \to \infty, (1 \leq r \leq 2)$. From (12), (41) and (43) and after similar simplification as in theorem 4.2 we have

$$\zeta_r(t,\lambda)\phi_s(a|x,\lambda) = \zeta_{Fr}(a|t,\lambda) + 0(|s|^{-1}[e^{|\rho|(x+t-2a)} + e^{|\rho|(x+b-t-a)}]). \tag{44}$$

as $|\lambda| \to \infty, (1 \leq r, s \leq 2)$. In [7], we showed that

$$w(\lambda) = w_F(\lambda) + 0(e^{|\rho|(b-a)}) \tag{45}$$

as $|\lambda| \to \infty$. Moreover $w_F(\lambda) = 0(|s|e^{|\rho|(b-a)})$ for large values of $|\lambda|$([7]). From (43), (44), (45) and by similar calculations as in theorem 4.1 we have

$$\frac{\zeta_r(t,\lambda)\phi_s(a|x,\lambda)}{w(\lambda)} = \frac{\zeta_{Fr}(t,\lambda)\phi_{Fs}(a|x,\lambda)}{w_F(\lambda)} + 0(|s|^{-2}[e^{|\rho|(x+t-b-a)} + e^{|\rho|(x-t)]}) \tag{46}$$

where $w_F(\lambda) \neq 0$, $w(\lambda) \neq 0$ and $(1 \leq r, s \leq 2)$ so that by (26) we have

$$\begin{aligned} G_1(x,t,\lambda) &= \frac{\zeta_2(t,\lambda)\phi_1(a|x,\lambda) + \zeta_1(t,\lambda)\phi_2(a|x,\lambda)}{w(\lambda)} && (47) \\ &= G_{F1}(x,t,\lambda) + 0(|s|^{-2}[e^{|\rho|(x+t-b-a)} + e^{|\rho|(x-t)}]) && (48) \end{aligned}$$

for suitably large values of $|\lambda|$. Similarly,

$$G_2(x,t,\lambda) = \frac{\zeta_2(t,\lambda)\phi_1(a|x,\lambda) - \zeta_1(t,\lambda)\phi_2(a|x,\lambda)}{w(\lambda)} \quad (49)$$

$$= G_{F2}(x,t,\lambda) + 0(|s|^{-2}[e^{|\rho|(b+a-x-t)} + e^{|\rho|(t-x)}]) \quad (50)$$

for suitably large values of $|\lambda|$. From theorem 4.1, (48) and (50) we have

$$G_1(x,t,\lambda) \sim G_{F1}(x,t,\lambda)$$

$$G_2(x,t,\lambda) \sim G_{F2}(x,t,\lambda)$$

for suitably large values of $|\lambda|$. Thus,

$$G(x,t,\lambda) \sim G_F(x,t,\lambda)$$

for suitably large values of $|\lambda|$ that is avoiding the eigenvalues where $w(\lambda) = 0$ and $w_F(\lambda) = 0$.

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