Approximating the Powers with Large Exponents and Bases Close to Unit, and the Associated Sequence of Nested Limits

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Abstract

The rate of convergence of \( \lim_{|t| \to \infty} (1 + \frac{x}{t})^t \) is estimated accurately and the associated sequence of nested limits starting with the limit

\[
\lim_{t \to \infty} \left[ t \cdot \left( \left( 1 + \frac{x}{t} \right)^t - e^x \right) \right] = -\frac{1}{2} e^x x^2
\]

is established.

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1 Introduction

In the mathematical literature we frequently meet the expressions like \( (1 + \frac{x}{t})^t \) which should be estimated for given \( x \in \mathbb{R} \) and large \( |t| \), see [1], for example. Moreover, in [3] the Wallis product

\[
W(n) := \prod_{k=1}^{n} \frac{4k^2}{4k^2 - 1}
\]

was presented in the form

\[
W(n) = \frac{1}{2} \pi e \left( 1 - \frac{1}{2n+1} \right)^{2n+1} \cdot \exp \left( \frac{\Theta_n}{6n} - \frac{\Theta'_n}{6n+3} \right)
\]
and consequently the expression

\[ \pi = W(n) \, e^{-1} \left( 1 + \frac{1}{2n} \right)^{2n+1} \cdot \exp \left( \frac{\Theta'_n}{6n+3} - \frac{\Theta_n}{6n} \right) \]

is given for every positive integer \( n \) with \( \Theta_n, \Theta'_n \in (0,1) \).

So, we are interested primarily how to estimate the power \( (1 + \frac{x}{t})^t \) for given \( x \in \mathbb{R} \) and large \( t \in \mathbb{R}^+ \). The rate of monotonous convergence of \( \lim_{t \to \infty} (1 + \frac{x}{t})^t \) was estimated in [2] as

\[ \exp \left( x - \frac{x^2}{2(1-\varepsilon)t} \right) < \left( 1 + \frac{x}{t} \right)^t < \exp \left( x - \frac{x^2}{2(1+\varepsilon)t} \right), \quad (1) \]

valid for every real \( x \neq 0, \varepsilon \in (0,1) \) and \( t \geq |x|/\varepsilon \). In the same paper the rate of convergence of

\[ \lim_{t \to \infty} \left[ t \cdot \left( e^x - \left( 1 + \frac{x}{t} \right)^t \right) \right] = \frac{e^x x^2}{2} \quad (2) \]

was also examined.

The formula (1) represents an approximations for the function \( t \mapsto (1 + \frac{x}{t})^t \), useful for very large \( t \). In what follows we shall establish more accurate approximations using only elementary techniques and we shall also answer the question

\[ \lim_{t \to \infty} \left\{ t \left[ \frac{e^x x^2}{2} - t \left( e^x - \left( 1 + \frac{x}{t} \right)^t \right) \right] \right\} = ?, \]

posed in [2]. In addition we shall find further nested limits starting with (2).

2 Preliminaries

For given \( x, t \in \mathbb{R} \setminus \{0\} \), such that \( |t| > |x| \), we have

\[ \left( 1 + \frac{x}{t} \right)^t = \exp \left( \varphi_x(h) \right) =: G(x, h), \quad (3) \]

where

\[ \varphi_x(h) \equiv \frac{1}{h} \ln(1 + xh) \quad (4) \]

with \( h = \frac{1}{t} \), \( x \) being a parameter. If \( 0 < |xh| < 1 \) the function \( \varphi_x(h) \) is well defined (on the punctured interval \((-1/|x|, 1/|x|))\) and we have, for any integer

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\[ m \geq 0, \]

\[ \varphi_x(h) = \frac{1}{h} \int_0^{xh} \frac{d\tau}{1 + \tau} = \frac{1}{h} \int_0^{xh} \left( \sum_{j=0}^{m} \left( -\tau \right)^j \left( \tau + m + 1 \right) \right) d\tau \]

\[ = S_m(x, h) + R_m(x, h), \quad (5) \]

where

\[ S_m(x, h) = \sum_{j=0}^{m} \frac{(-1)^j x^{j+1}}{j+1} h^j, \quad (6) \]

\[ R_m(x, h) = \frac{(-1)^{m+1}}{h} \int_0^{xh} \tau^{m+1} d\tau. \quad (7) \]

For the remainder \( R_m(x, h) \) we have, if \( 0 < |xh| < 1 \),

\[ |R_m(x, h)| \leq \frac{1}{|h|} \int_0^{|xh|} \frac{s^{m+1} ds}{1 - |xh|} = \frac{|xh|^{m+2}}{|h|(m+2)(1 - |xh|)} \rightarrow 0, \quad \text{as} \quad m \rightarrow \infty. \quad (8) \]

Hence,

\[ \varphi_x(h) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{j+1}}{j+1} h^j, \quad (9) \]

for \( 0 < |h| < 1/|x| \). Therefore, \( \varphi_x(h) \) is represented by a function which is analytic on the disk \( |h| < 1/|x| \). Additionally, we define \( \varphi_x(0) = x \) (the sum of the series \( (9) \)) and \( G(x, 0) = e^x \). Consequently, \( \varphi_x \in C^\infty(-1/|x|, 1/|x|) \) and, referring to \( (3) \), the same is true for the function \( h \mapsto \exp\left( \varphi_x(h) \right) \). Hence, for \( G \) considered as the function of the variable \( h \), we have

\[ G \in C^\infty(-1/|x|, 1/|x|), \quad G(x, h) = (1 + xh)^{1/h} \text{ for } h \neq 0, \quad \text{and} \quad G(x, 0) = e^x. \quad (10) \]

According to \( (9) \), the derivatives are given as

\[ \varphi_x^{(j)}(0) = \frac{(-1)^j j!}{j+1} x^{j+1}, \quad (11) \]
for \( j \geq 1 \). Consequently, we obtain

\[
\frac{\partial G}{\partial h}(x, 0) = -\frac{1}{2} e^x x^2, \quad \frac{\partial^2 G}{\partial h^2}(x, 0) = \frac{1}{12} e^x (3x^4 + 8x^3)
\]

\[
\frac{\partial^3 G}{\partial h^3}(x, 0) = -\frac{1}{8} e^x (x^6 + 8x^5 + 12x^4)
\]

\[
\frac{\partial^4 G}{\partial h^4}(x, 0) = \frac{1}{240} e^x (15x^8 + 240x^7 + 1040x^6 + 1152x^5)
\]

\[
\frac{\partial^5 G}{\partial h^5}(x, 0) = -\frac{1}{96} e^x (3x^{10} + 80x^9 + 680x^8 + 2112x^7 + 1920x^6),
\]

for example.

In particular cases, considering the sign of the product \( xh \), the remainder \( R_m(x, h) \) in (7) can be estimated more precisely. We distinguish two possibilities: \( xh > 0 \) and \( xh < 0 \).

A) \([0 < xh < 1]\) In this case we estimate

\[
\int_0^{xh} \frac{\tau^{m+1}}{1 + \tau} d\tau > \int_0^{xh} \frac{\tau^{m+1}}{1 + xh} d\tau = \frac{(xh)^{m+2}}{(m + 2)(1 + xh)}
\]

\[
\int_0^{xh} \frac{\tau^{m+1}}{1 + \tau} < \int_0^{xh} \frac{\tau^{m+1}}{1 + \tau} d\tau = \frac{(xh)^{m+2}}{m + 2}.
\]

Consequently, using (7), we obtain

\[
\frac{(xh)^{n+2}}{(m + 2)(1 + xh)|h|} < (-1)^{m+1} \text{sgn}(h) \cdot R_m(x, h) < \frac{(xh)^{m+2}}{(m + 2)|h|}.
\]  \( (13) \)

B) \([-1 < xh < 0]\) Now we have, referring to (7),

\[
R_m(x, h) = \frac{(-1)^{m+1}}{h} \int_0^{-xh} \frac{(-s)^{m+1}}{1 - s} (-ds) = -\frac{1}{h} \int_0^{[xh]} s^{m+1} \frac{ds}{1 - s}.
\]

Therefore, proceeding as above, we find

\[
\frac{|xh|^{m+2}}{(m + 2)|h|} < \text{sgn}(-h) \cdot R_m(x, h) < \frac{|xh|^{m+2}}{(m + 2)(1 - |xh|)|h|}.
\]  \( (14) \)

### 3 Approximating the function \( t \mapsto \left(1 + \frac{x}{t}\right)^t \)

How to estimate the expression \( (1 + \frac{x}{t})^t \) answers the following theorem.
Theorem 1. For any integer \( m \geq 0 \), for \( x \in \mathbb{R} \setminus \{0\} \) and for \( t > |x| \) we have the equality

\[
\left(1 + \frac{x}{t}\right)^t = e^x \cdot \exp\left(s_m(x, t) + r_m(x, t)\right),
\]

(15)

where\(^1\)

\[
s_m(x, t) = \sum_{j=1}^{m} (-1)^j \frac{x^{j+1}}{j+1} t^{-j} = -\sum_{k=2}^{m+1} (-1)^k \frac{x^k}{k} t^{-k+1}
\]

(16)

and the remainder \( r_m(x, t) \) is estimated in the following way:

\[
(1) \quad \frac{x^{m+2}}{(m+2)(t+|x|)^m} < (-1)^{m+1} r_m(x, t) < \frac{x^{m+2}}{(m+2) t^{m+1}}, \quad \text{if } x > 0,
\]

(17)

\[
(2) \quad -\frac{|x|^{m+2}}{(m+2)(t-|x|)^m} < r_m(x, t) < -\frac{|x|^{m+2}}{(m+2) t^{m+1}}, \quad \text{for } x < 0.
\]

(18)

Proof. Let \( m, x \) and \( t \) fulfill all the conditions of the theorem. Then, considering (3), and substituting \( h = \frac{1}{t} \) in (5), we obtain the expression (15) satisfying (16). Then, using (13), we find the estimate (17) and, referring to (14), we get the inequalities (18).

Setting \( m = 0, 1, 2 \) in the preceding theorem, we obtain the following two corollaries.

Corollary 1.1. For \( x > 0 \) and \( t > x \) the following inequalities hold

\[
e^x \cdot \exp\left(-\frac{x^2}{2t}\right) < (1 + \frac{x}{t})^t < e^x \cdot \exp\left(-\frac{x^2}{2(t+x)}\right),
\]

\[
e^x \cdot \exp\left(-\frac{x^2}{2t} + \frac{x^3}{3t(t+x)}\right) < (1 + \frac{x}{t})^t < e^x \cdot \exp\left(-\frac{x^2}{2t} + \frac{x^3}{3t^2}\right),
\]

\[
e^x \cdot \exp\left(-\frac{x^2}{2t} + \frac{x^3}{3t^2} - \frac{x^4}{4!t^2}\right) < (1 + \frac{x}{t})^t < e^x \cdot \exp\left(-\frac{x^2}{2t} + \frac{x^3}{3t^2} - \frac{x^4}{4!t^2}\right).
\]

Corollary 1.2. For \( x < 0 \) and \( t > |x| \) we have the inequalities

\[
e^x \cdot \exp\left(-\frac{x^2}{2(t-|x|)}\right) < (1 + \frac{x}{t})^t < e^x \cdot \exp\left(-\frac{x^2}{2t}\right),
\]

\[
e^x \cdot \exp\left(-\frac{x^2}{2t} - \frac{|x|^3}{3t(t-|x|)}\right) < (1 + \frac{x}{t})^t < e^x \cdot \exp\left(-\frac{x^2}{2t} - \frac{|x|^3}{3t^2}\right),
\]

\[
e^x \cdot \exp\left(-\frac{x^2}{2t} - \frac{|x|^3}{3t^2} - \frac{x^4}{4!t^2}\right) < (1 + \frac{x}{t})^t < e^x \cdot \exp\left(-\frac{x^2}{2t} - \frac{|x|^3}{3t^2} - \frac{x^4}{4!t^2}\right).
\]

\(^1\)By definition, \( \sum_{j=p}^q = 0 \), for \( p > q \).
Consequently, inserting \( x = ht \) in the corollaries above we get the following corollary.

**Corollary 1.3.** For \( t > 0 \) and \( |h| < 1 \) we have the estimates

\[
e^{ht} \cdot \exp \left( -\frac{h^2 t}{2} \right) < (1 + h)^t < e^{ht} \cdot \exp \left( -\frac{h^2 t}{2(1 + h)} \right), \quad \text{for } 0 < h < 1,
\]
\[
e^{ht} \cdot \exp \left( -\frac{h^2 t}{2(1 - |h|)} \right) < (1 + h)^t < e^{ht} \cdot \exp \left( -\frac{h^2 t}{2} \right), \quad \text{for } -1 < h < 0.
\]

**Example.** Using the Corollaries 1.1 and 1.2, we obtain, for \( t > 1 \), the following estimates:

\[
e \cdot \exp \left( -\frac{1}{2t} \right) < \left( 1 + \frac{1}{t} \right)^t < e \cdot \exp \left( -\frac{1}{2(t + 1)} \right),
\]
\[
\frac{1}{e} \cdot \exp \left( -\frac{1}{2(t - 1)} \right) < \left( 1 - \frac{1}{t} \right)^t < \frac{1}{e} \cdot \exp \left( -\frac{1}{2t} \right),
\]
\[
\sqrt{e} \cdot \exp \left( -\frac{1}{8t} \right) < \left( \frac{2t + 1}{2t} \right)^t < \sqrt{e} \cdot \exp \left( -\frac{1}{8t + 4} \right),
\]
\[
\frac{1}{\sqrt{e}} \cdot \exp \left( \frac{1}{8t + 4} \right) < \left( \frac{2t}{2t + 1} \right)^t < \frac{1}{\sqrt{e}} \cdot \exp \left( \frac{1}{8t} \right).
\]

From the second inequality it also follows

\[
e \cdot \exp \left( \frac{1}{2t} \right) < \left( 1 - \frac{1}{t} \right)^{-t} < e \cdot \exp \left( \frac{1}{2(t - 1)} \right), \quad \text{for } t > 1.
\]

**Corollary 1.4.** For \( t > x > 0 \) there holds the estimate

\[
e^x > e^* (x, t) := \left( 1 + \frac{x}{t} \right)^t \left[ 1 + \frac{x^2}{2t} - \frac{x^3}{3!t^2} + \frac{x^4}{4!(t + x)t^2} \right]
\]
\[
e^x < e^{**} (x, t) := \left( 1 + \frac{x}{t} \right)^t \left[ 1 + 2 \frac{x^2}{2t} \left( \frac{x^2}{2t} - \frac{x^3}{3!t^2} + \frac{x^4}{4t^3} \right) \right].
\]

**Proof.** Using Theorem 1 with \( m = 2 \), we have

\[
e^x = \left( 1 + \frac{x}{t} \right)^t \exp \left( -s_2(x, t) - r_2(x, t) \right)
\]

where \( -s_2(x, t) = \frac{x^2}{2t} - \frac{x^3}{3!t^2} \).
and
\[
\frac{x^4}{4(t + x)t^2} < -r_2(x, t) < \frac{x^4}{4t^3}.
\]
Thus, for \( t > x > 0 \),
\[
-s_2(x, t) - r_2(x, t) > \frac{x^2}{2t} - \frac{x^3}{3t^2} + \frac{x^4}{4(t + x)t^2} > 0
\]
and
\[
-s_2(x, t) - r_2(x, t) < \frac{x^2}{2t} - \frac{x^3}{3t^2} + \frac{x^4}{4t^3}
\]
\[
= \frac{x}{t} \left(\frac{x}{2} - \frac{x}{3} \left(\frac{x}{t}\right) + \frac{x}{4} \left(\frac{x}{t}\right)^2\right)
\]
\[
< \frac{x}{t} \left(\frac{x}{2} - \frac{x}{3} \left(\frac{x}{t}\right) + \frac{x}{4} \left(\frac{x}{t}\right)\right)
\]
\[
< \frac{x}{t} \cdot \frac{x}{2}.
\]
These estimates, together with the expression (19) and the inequalities \( 1 + h < e^h < 1 + e^h \cdot h < 1 + 4^h \cdot h \), valid for \( h > 0 \), verify the corollary. ■

Figure 1 illustrates the rational approximations to \( e^x \), given in Corollary 1.4, for \( x = 1 \).

Figure 1: The graphs of the sequences \( n \mapsto e^*(1, n) \) and \( n \mapsto e^{**}(1, n) \); Corollary 1.4.

\section{4 Nested limits}

For the function
\[
t \mapsto \left(1 + \frac{x}{t}\right)^t =: E(x, t),
\]
$x \in \mathbb{R}$ being a parameter, we define $E_1(x,t)$, $E_2(x,t)$, etc... by

\[ E_1(x,t) := t[E(x,t) - e^x], \quad e_1(x) = \lim_{t \to \infty} E_1(x,t) = -\frac{e^x x^2}{2}, \]
\[ E_2(x,t) := t[E_1(x,t) - e_1(x)], \quad e_2(x) := \lim_{t \to \infty} E_2(x,t), \quad \text{etc...} \]

Here we have two questions about the sequence $(e_n(x))_{n \in \mathbb{N}}$: its existence and its computation. In order to solve this problem we shall use the following lemma.

**Lemma 1.** Let $r \in \mathbb{R}^+$, the interval $I = (-r, r)$, $f \in C^\infty(I)$,

\[
 f_1(h) = \begin{cases} 
 f'(0), & h = 0, \\
 \frac{f(h) - f(0)}{h}, & 0 \neq |h| < r 
\end{cases}
\]  

(21)

and, for any $n \in \mathbb{N}$,

\[
 f_{n+1}(h) = \begin{cases} 
 f^{(n+1)}(0), & h = 0, \\
 \frac{f_n(h) - f_n(0)}{h}, & 0 \neq |h| < r.
\end{cases}
\]  

(22)

Then the sequence $(f_n(h))_{n \in \mathbb{N}}$ is well defined at any $h \in I$. Moreover, for every $n \in \mathbb{N}$ we have

\[
 f_n(h) = \begin{cases} 
 \frac{f^{(n)}(0)}{(n)!}, & h = 0, \\
 h^{-n} \left(f(h) - \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{i!} h^i\right), & 0 \neq |h| < r,
\end{cases}
\]  

(23)

with $f_n$ differentiable at $h = 0$ and

\[
 f_{n+1}(0) = f'_n(0) = \frac{f^{(n+1)}(0)}{(n+1)!}. 
\]  

(24)

**Proof.** We can verify the equalities (23) by induction. Then, considering (23) and using the Taylor formula of order $n$, we have, for any $n \in \mathbb{N}$ and $h \in I \setminus \{0\}$,

\[
 f_{n+1}(h) = \frac{f_n(h) - f_n(0)}{h} = \frac{f^{(n+1)}(\vartheta h)}{(n+1)!},
\]

for some $\vartheta = \vartheta(n,h) \in (0,1)$. Since $f \in C^\infty(I)$ we get

\[
 f_{n+1}(0) = f'_n(0) = \lim_{h \to 0} \frac{f_n(h) - f_n(0)}{h} = \frac{f^{(n+1)}(0)}{(n+1)!}.
\]
Now we are in a position to formulate the following theorem about the nested limits.

**Theorem 2.** For any \( n \in \mathbb{N}, x \in \mathbb{R} \setminus \{0\} \) and \( |t| > |x| \), let \( E(x,t) := (1 + \frac{x}{t})^t \) and \n\[E_1(x,t) := t[E(x,t) - e^x], \quad e_n(x) := \lim_{|t| \to \infty} E_n(x,t), \quad E_{n+1}(x,t) := t[E_n(x,t) - e_n(x)].\]

Then the sequences \((E_n(x,t))_{n \in \mathbb{N}}\) and \((e_n(x))_{n \in \mathbb{N}}\) are well defined for any \( |t| > |x| \) and we have (see (3) and (12))

\[e_n(x) = \frac{1}{n!} \cdot \frac{\partial^n G}{\partial x^n} (x,0), \quad \text{for } n \in \mathbb{N} \text{ and } x \in \mathbb{R} \setminus \{0\}.\]

**Proof.** We fix \( x \in \mathbb{R} \setminus \{0\} \) determining the interval \( I = (-r,r) \) with \( r = 1/|x| \). Next, we consider the function \( f, f(h) \equiv G(x,h) \). According to (10), we have \( f \in C^\infty(I) \). Referring to Lemma 1, \( f \) generates the sequence \((f_n(h))_{n \in \mathbb{N}}\) such that the relations (21), (22) and (24) hold.

Referring to (10) and (21), we obtain, for \( |t| > |x| \),

\[E_1(x,t) = t(E(x,t) - e^x) = t(G(x,1/t) - G(x,0)) = \frac{f(1/t) - f(0)}{1/t} = f_1(1/t)\]

and, defining \( f_0(h) := f(h) \),

\[e_1(x) = \lim_{|t| \to \infty} E_1(x,t) = f_1(0) = f'(0) = f_0'(0).\]

Generally, for \( n \in \mathbb{N} \) and \( |t| > |x| \), we have

\[E_n(x,t) = f_n(1/t) \quad \text{and} \quad e_n(x) = f_n(0) = f_{n-1}'(0). \quad (25)\]

Indeed, if (25) is true, then

\[E_{n+1}(x,t) = t[E_n(x,t) - e_n(x)] = \frac{f_n(1/t) - f_n(0)}{1/t} = f_{n+1}(1/t).\]

Consequently, considering the continuity (differentiability) of \( f_n \),

\[e_{n+1}(x) = \lim_{|t| \to \infty} E_{n+1}(x,t) = \lim_{|t| \to \infty} f_{n+1}(1/t) = f_{n+1}(0) = f_n'(0).\]

Hence, the sequences \((E_n(x,t))_{n \in \mathbb{N}}\) and \((e_n(x))_{n \in \mathbb{N}}\) are well defined for any \( |t| > |x| > 0 \) and, referring to (25) and (24), we get

\[e_n(x) = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \cdot \frac{\partial^n G}{\partial x^n} (x,0).\]

\[\blacksquare\]
From Theorem 2 and relations (12) the following Corollary follows.

**Corollary 2.1.** For \( x \neq 0 \) we have

\[
\begin{align*}
    e_1(x) &= -\frac{1}{2} e^x x^2, \\
    e_2(x) &= \frac{1}{24} e^x \left(3x^4 + 8x^3\right), \\
    e_3(x) &= -\frac{1}{48} e^x \left(x^6 + 8x^5 + 12x^4\right), \\
    e_4(x) &= \frac{1}{5760} e^x \left(15x^8 + 240x^7 + 1040x^6 + 1152x^5\right), \\
    e_5(x) &= -\frac{1}{11520} e^x \left(3x^{10} + 80x^9 + 680x^8 + 2112x^7 + 1920x^6\right).
\end{align*}
\]

Figure 2 shows the graphs of the functions \( x \mapsto e_n(x) \) from Corollary 2.1.

![Graphs of e_n(x)](image)

**Figure 2:** The graphs of the functions \( x \mapsto e_n(x) \).

**Open question.** Find the closed-form formula for the sequence \( n \mapsto e_n(x) \), i.e. find the closed-form formula for the sequence \( n \mapsto P_n(x) \) of polynomials such that \( e_n(x) \equiv e^x P_n(x) \).

**References**


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