Factorable Matrix Transforms of Summability
Domains of Cesàro Matrices

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Abstract

In this paper some classes of triangular factorable matrices, transforming the summability domain of Cesàro matrix into the summability domain of a matrix $B$ with real or complex entries, are described.

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1 Introduction

In the present paper the transforms of summability domains of Cesàro matrices by triangular factorable matrices are studied. Let $A = (a_{nk})$ be a matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to $\infty$ unless otherwise specified. A sequence $x := (x_k)$ or a series $x := \sum_k x_k$ is said to be $A$-summable if the sequence $A_n x := (A_n x_k)$ is convergent, where

$$A_n x := \sum_k a_{nk} x_k.$$ 

Let

$$c := \left\{ x = (x_k) \mid \exists \lim k x_k \right\}, \quad cs := \left\{ x = (x_k) \mid \exists \lim n \sum_{k=0}^n x_k \right\}$$

$$l := \left\{ x = (x_k) \mid \sum_k |x_k| < \infty \right\}, \quad c_A := \left\{ x = (x_k) \mid Ax \in c \right\}.$$

A matrix $A$ is called sequence-to-sequence conservative (shortly, $Sq-Sq$ conservative) if $Ax \in c$ for each $x \in c$. If $Ax \in c$ for each $x \in cs$, then a matrix
A is called series-to-sequence conservative (shortly, Sr-Sq conservative). A matrix $A$ is said to be series-to-sequence regular (shortly, Sr-Sq regular) if
\[ \lim_{n \to \infty} A_n x = \lim_{n \to \infty} \sum_{k=0}^{n} x_k \text{ for every } x \in cs. \]

Let $\mathcal{M}$ be the set of all lower triangular factorable matrices $M = (m_{nk})$, where
\[ m_{nk} = r_n v_k, \quad k \leq n; \quad r_n, v_k \in \mathcal{C}, \]

Let $C^\alpha = (a_{nk})$, $\alpha \in \mathcal{C}\{−1, −2, ...\}$, be a series-to-sequence Cesàro matrix, i.e. (see [4] or [5])
\[
a_{nk} := \begin{cases} \frac{A^\alpha_{n-k}}{A_n} & (k \leq n), \\ 0 & (k > n), \end{cases}
\]

where $A^\alpha_n = \binom{n+\alpha}{n}$ are Cesàro numbers. In [1] and [8] necessary and sufficient conditions for a matrix $M$ with real or complex entries to be a transform from $c_{C^\alpha}$ into $c_B$ for certain $\alpha \in \mathcal{C}$ and certain triangular matrix $B$ are described. Moreover, in [3] this problem is considered for the special case $B = C^\beta$, and in [2] one class of triangular matrices $M$, transforming $c_{C^\alpha}$ into $c_{C^\beta}$, is described.

In the present paper some classes of triangular factorable matrices $M$, transforming $c_{C^\alpha}$ into $c_{C^\beta}$, are described. The paper is organized as follows. In Section 2 some auxiliary results are presented, which are needed later. In Section 3 sufficient conditions for $M \in \mathcal{M}$ to be a transform from $c_A$ into $c_B$ are found. In Section 4 some classes of triangular factorable matrices $M$ from $\mathcal{M}$, transforming $c_{C^\alpha}$ into $c_B$ are described.

2 Auxiliary results

In this section we present some auxiliary results, which we need further.

**Lemma 2.1** (cf. [5], p. 46-47). A matrix $D = (d_{nk})$ is Sq-Sq conservative if and only if
\[
\text{there exist finite limits } \lim_n d_{nk} = d_k, \quad (1)
\]
\[
\text{there exist finite limits } \lim_n \sum_k d_{nk} = d, \quad (2)
\]
\[
\sum_k |d_{nk}| = O(1). \quad (3)
\]

Also we need the following properties of Cesàro numbers (see [4], p. 77-81):
\[
\sum_{n=k}^{\infty} \frac{A_n^\alpha}{A_n^\beta} = \frac{\beta}{\beta - \alpha - 1} \frac{1}{A_k^{\beta-\alpha-1}} \text{ for } Re\beta \geq 0, Re(\beta - \alpha) > 1, k = 1, 2, ..., \quad (4)
\]
\[
|A_n^\alpha| \geq L(n+1)^{Re\alpha} \text{ for } \alpha \in \mathcal{C}\{−1, −2, ...\}, \quad L > 0. \quad (5)
\]
Lemma 2.2 (cf. [4], p. 192). Let $\alpha \in \mathbb{C}$ with $\text{Re}\alpha > 0$ or $\alpha = 0$, and $(v_k)$ is a sequence of complex numbers. A series $\sum_k v_k x_k$ is convergent for each $\sum_k x_k \in c_{\mathbb{C}\alpha}$ if and only if

$$v_k = \mathcal{O} [(k + 1)^{-\text{Re}\alpha}]$$

(6) and

$$\sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} |\Delta_{k+1}^{\alpha} v_k| = \mathcal{O}(1),$$

(7) where

$$\Delta_{k+1}^{\alpha} v_k := \sum_{n=k}^{\infty} A_{n-k}^{-\alpha-2} v_n.$$

3 Matrix transforms from $c_A$ into $c_B$

At first we give a simple necessary condition for $M \in \mathcal{M}$ to be a transform from $c_A$ into $c_B$.

**Proposition 3.1** Let $A = (a_{nk})$ be a matrix with $e^0 = (1, 0, 0, \ldots) \in c_A$ and $B = (b_{nk})$ an arbitrary matrix with real or complex entries. If $M = (r_nv_k) \in \mathcal{M}$ transforms $c_A$ into $c_B$, then $(r_n) \in c_B$.

**Proof** easily follows from the relation

$$M_n e^0 = r_n v_0.$$

Now we present sufficient conditions for $M \in \mathcal{M}$ to be a transform from $c_A$ into $c_B$.

**Theorem 3.2** Let $A = (a_{nk})$ and $B = (b_{nk})$ be matrices with real or complex entries, $(r_n)$ and $(v_k)$ sequences with real or complex entries and $B^t = (b^t_{pn})$ a matrix, defined by the relation $b^t_{pn} = b_{pn} r_n$. Then $M = (r_nv_k) \in \mathcal{M}$ transforms $c_A$ into $c_B$ if

$$(v_k x_k) \in c_s \text{ for every } x \in c_A,$$

(8)

$B^t$ is $\text{Sq} - \text{Sq}$ conservative.

(9)

**Proof** easily follows from the equality

$$\sum_n b_{pn} M_n x = \sum_n b^t_{pn} \sum_{k=0}^{n} v_k x_k$$

for each $x \in c_A$. 

Proposition 3.3 Let $B = (b_{nk})$ be a Sr-Sq regular matrix, where $b_{nk} > 0$ for all $n$ and $k$, and $(r_n)$ a sequence with real or complex entries. Then condition (9) is satisfied, i.e. $B^t = (b_{pn}^t) = (b_{pn}r_n)$ is Sq-Sq conservative if and only if $(r_n) \in l$.

Proof. Necessity. We suppose that $B^t$ is Sq-Sq conservative and show that then $(r_n) \in l$. Indeed, condition (3) of Lemma 2.1 takes for $D = B^t$ the form

$$T_p := \sum_n |b_{pn}r_n| = \sum_n b_{pn}|r_n| = O(1).$$

(10)

If $\sum_n |r_n| = \infty$, then (see [6], p. 92) $\lim_{p \to \infty} T_p = \infty$, i.e. condition (10) is not satisfied. Hence $(r_n) \in l$ by Lemma 1.

Sufficiency. Let $(r_n) \in l$. We show that all conditions of Lemma 2.1 are fulfilled for $D = B^t$. Indeed, the Sr-Sq regularity of $B$ implies that $(r_n) \in c_B$, i.e. condition (2) of Lemma 2.1 is satisfied for $D = B^t$. The Sr-Sq regularity of $B$ also implies that $b_{nk} = O(1)$ and there exist the finite limits $\lim_n b_{nk}$ by Proposition 17 of [7]. Consequently condition (1) is fulfilled for $D = B^t$, and

$$T_p = O(1) \sum_n |r_n| = O(1),$$

i.e. condition (3) of Lemma 2.1 is satisfied for $D = B^t$. Therefore $B^t$ is Sq-Sq conservative by Lemma 2.1.

Remark. The assertion of Proposition 3.3 holds also for lower triangular matrix $B = (b_{nk})$, where $b_{nk} > 0$ for all $k \leq n$.

Theorem 3.4 Let $A = (a_{nk})$, $B = (b_{nk})$ be matrices with real or complex entries and $(r_n)$, $(v_k)$ sequences with real or complex entries. Moreover, let $l \subset c_B$ and $(r_n) \in l$. Then $M = (r_nv_k) \in \mathcal{M}$ transforms $c_A$ into $c_B$ if condition (8) is fulfilled.

Proof. Let

$$S_n := \sum_{k=0}^n v_k x_k$$

for every $x \in c_A$. As $(S_n) \in c$ for every $x \in c_A$ by (8), then $(S_n)$ is also bounded for each $x \in c_A$. Therefore

$$\sum_n |M_n x| = \sum_n |r_n S_n| = O(1) \sum_n |r_n| = O(1)$$

for every $x \in c_A$. As $l \subset c_B$, then $M$ transforms $c_A$ into $c_B$. 
Factorable matrix transforms

4 Matrix transforms from $cC^\alpha$ into $c_B$

In this section we consider the factorable matrix transforms of summability domains of Cesàro matrices.

Theorem 4.1 Let $\alpha \in \mathbb{C}$ with $\text{Re}\alpha > 0$ or $\alpha = 0$, and $B = (b_{nk})$ be a matrix with the property $l \subset c_B$. Let $(v_k)$ be defined by $v_k := 1/A_k^t$, where $t \in \mathbb{C}$ with $\text{Ret} > 0$, and $(r_n) \in l$. Then $M = (r_nv_k) \in \mathcal{M}$ transforms $cC^\alpha$ into $c_B$ if $\text{Re}\alpha \leq \text{Ret}$.

Proof. By Theorem 3.4 it is sufficient to show that condition (8) is fulfilled for $A = C^\alpha$ and $v_k = 1/A_k^t$. With the help of (4) and (5) we have

$$
\sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} \left| \Delta_k^{\alpha+1} v_k \right| = \sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} \left| \sum_{n=k}^{\infty} A_{n-k}^{-\alpha-2} A_n^t \right|
$$

$$
= \sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} \frac{t}{t + \alpha + 1} A_k^{-\alpha+1} = \mathcal{O}(1) \sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} \frac{1}{(k + 1)^{\text{Re}(t+\alpha)+1}}
$$

$$
= \mathcal{O}(1) \sum_{k=0}^{\infty} \frac{1}{(k + 1)^{\text{Ret}+1}} = \mathcal{O}(1),
$$

since $\text{Ret} > 0$, i.e. condition (7) is satisfied. Condition (6) is also fulfilled, since by (5) there exists $L > 0$ so that

$$
\left| \frac{1}{A_k^t} \right| \leq \frac{1}{L(k + 1)^{\text{Ret}}} = \mathcal{O}(1)k + 1)^{-\text{Ret}} = \mathcal{O}(1)k + 1)^{-\text{Re}\alpha}.
$$

Consequently condition (8) is fulfilled by Lemma 2.2. Thus $M$ transforms $cC^\alpha$ into $c_B$ by Theorem 3.4.

Theorem 4.2 Let $\alpha \in \mathbb{C}$ with $\text{Re}\alpha > 0$ or $\alpha = 0$, and $B = (b_{nk})$ be a matrix with the property $l \subset c_B$. Let $(v_k)$ be defined by $v_k := y^k$, where $y \in \mathbb{C}$, and $(r_n) \in l$. Then $M = (r_nv_k) \in \mathcal{M}$ transforms $cC^\alpha$ into $c_B$ if $|y| < 1$.

Proof. By Theorem 3.4 it is sufficient to show that condition (8) is fulfilled for $A = C^\alpha$ and $v_k = y^k$. As

$$
\sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} \left| \Delta_k^{\alpha+1} v_k \right| = \sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} \left| \sum_{n=0}^{\infty} A_{n}^{-\alpha-2} y^{n+k} \right|
$$

$$
\leq \sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} y^k \sum_{n=0}^{\infty} A_{n}^{-\alpha-2} y^n = \mathcal{O}(1) \sum_{k=0}^{\infty} (k + 1)^{\text{Re}\alpha} y^k < \infty
$$
(since the series $\sum_{k=0}^{\infty}(k+1)^{Re\alpha}y^k$ converges by the convergence criterion of Cauchy for positive series), then condition (7) is satisfied. Also condition (6) is fulfilled, since

$$\lim_k y^k(k+1)^{Re\alpha} = 0.$$  

Consequently condition (8) is fulfilled. Thus $M$ transforms $c_{C^\alpha}$ into $c_B$ by Theorem 3.4.

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References


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