Fixed Points on L-Embedded Banach Spaces

Reeja P. V. and K. T. Ravindran

P G Department of Mathematics
Payyanur College, Payyanur, India
pvreeja@gmail.com
drravindran@gmail.com

Abstract

In this paper we are investigating about fixed points for mappings on the unit ball of an L-Embedded Banach Space endowed with abstract measure topology and $(P)_\tau$-fixed point property of an asymptotic contractive map on a subset of this space.

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1 Introduction

An L-embedded Banach space is a Banach space which can be complemented in its bidual, in which the norm is additive between the space and its complement. The interest of L-embeddedness lies in the fact that it unifies some Banach space properties which before were known only as special cases. A kind of break-through for this general point of view was Godefroy’s discovery that L-embedded Banach spaces are weakly sequentially complete. Every L-Embedded Banach space can be endowed with the so-called abstract measure topology, which is defined by specifying the class of convergent sequence. This topology is equivalent to the usual measure topology on bounded sets when $X$ is the pre-dual of a finite von Neumann algebra. In the case of L-embedded space unit ball plays a crucial role, as a norm closed subspace is L-embedded if and only if its unit ball is closed in measure.
2 Preliminaries

Definition 2.1 (L-Projection)[3]: A linear projection $P$ on a Banach space $X$ is called an L-projection if $||x|| = ||Px|| + ||x - Px||$ for all $x \in X$.

Definition 2.2 (L-embedded Banach space)[3]: An L-embedded Banach space is a Banach space if it is the image of an L-projection on its bidual. Examples of L-embedded spaces include $l^1(\Gamma) = (C_0(\Gamma))^*$ (if any set), $N(H) = (K(H))^*$ (the nuclear operators on a Hilbert space $H$).

Definition 2.3 (Abstract measure topology)[7]: Let $X$ be a Banach space. A system $\tau_\mu$ of subsets of $X$ is called an abstract measure topology if it satisfies

1. $(X, \tau_\mu)$ is a sequential space in which every convergent sequence has a unique limit.
2. $\tau_\mu$ is weaker than the norm topology.
3. $\tau_\mu$ is translation invariant for sequences more precisely, $x_n \xrightarrow{\tau_\mu} x$ if and only if $x_n - x \xrightarrow{\tau_\mu} 0$ for any sequence $(x_n)$ in $X$.
4. Each bounded sequence in $X$ that spans $l^1$ asymptotically $\tau_\mu$-converges to 0, and each sequence in $X$ that $\tau_\mu$-converges to 0 is bounded and contains a subsequence which spans $l^1$ asymptotically or tends to 0 in norm.

Theorem 2.4 [7] Every L-embedded Banach space admits an abstract measure topology.

Theorem 2.5 [7] Let $X$ be an L-embedded Banach space with its abstract measure topology $\tau_\mu$. Then the following statements hold. (a) A sequence converges in norm if and only if it converges both weakly and with respect to $\tau_\mu$, and all limits coincide. (b) A norm closed subspace $Y \subset X$ is reflexive if and only if $\tau_\mu$ and the norm topology coincide on the unit ball of $Y$.

Theorem 2.6 [7] Let $X$ be an L-embedded Banach space endowed with its abstract measure topology $\tau_\mu$. Then a norm closed subspace $Y \subset X$ is L-embedded if and only if its unit ball $B_Y$ is $\tau_\mu$-closed.

Theorem 2.7 [6] If $K$ is a bounded closed convex subset of a Banach space and if $T : K \rightarrow K$ is nonexpansive, then there exists a sequence $(x_n) \subset K$ such that $\lim_{n \rightarrow \infty} ||x_n - T(x_n)|| \subset K = 0$.

Theorem 2.8 [1] Let $X$ be a normed linear space and $Y$ a subspace of $X$. Then set of best approximations to $x$ from $Y$, $P_Y(x)$ is convex.
Theorem 2.9 [3] Let X be a Banach space which is an L-summand in its bidual and Y a subspace of X, which is also an L-summand in its bidual. Then Y is proximinal in X and \( P_Y(x) \), the set of best approximations to x from Y, is weakly compact for all \( x \in X \).

Definition 2.10 (Contraction mapping) [6]: Contraction mapping, or contraction, on a normed space X is a self mapping \( f \) with the property that there is some real number \( k < 1 \) such that for all \( x \) and \( y \) in X,
\[
\| f(x) - f(y) \| \leq k \| x - y \|
\]

Theorem 2.11 [6] Banach fixed point theorem states that every contraction mapping on a nonempty Banach space has a unique fixed point.

Definition 2.12 (Non expansive mapping) [7]: A self mapping \( f \) on a Banach space is called non expansive mapping if there is some real number \( k \leq 1 \) such that for all \( x \) and \( y \) in X,
\[
\| f(x) - f(y) \| \leq k \| x - y \|
\]

Definition 2.13 [6]: Let X be a Banach space and \( D \subset X \). Then,
\[
diam(D) = \sup\{\| u - v \|: u, v \in D \}; \quad r_x(D) = \sup\{\| x - v \|: v \in D \}
\]
\[
r(D) = \inf\{r_x(D): x \in D\}; \quad C(D) = \{z \in D: r_z(D) = r(D)\}
\]

The number \( r(D) \) and the set \( C(D) \) are called the Chebyshev radius and Chebyshev centre of \( D \) respectively.

Definition 2.14 (Normal structure) [6]: A Banach Space X or a closed convex subset \( K \) of X, has normal structure if any bounded convex subset \( H \) of \( K \) which contains more than one point contains a non-diametral point, that is there exists a point \( x_0 \in H \) such that \( \sup\{\| x_0 - x \|: x \in H \} < \diam(H) = \sup\{\| x - y \|: x, y \in H \} \).

Definition 2.15 (\( \tau \)-SLSC) [4]: Let X be a Banach space and \( \tau \) be a vector topology on X. A function \( f: X \to R \) is sequentially lower semi-continuous with respect to \( \tau \) (\( \tau \)-SLSC) if \( f(x) \leq \limsup_{n \to \infty} f(x_n) \) for every sequence \( (x_n) \) in X such that \( (x_n) \) converges to x.

Definition 2.16 (Kadec-Klee property) [4]: The space X has Kadec-Klee property with respect to \( \tau \) (\( KK(\tau) \)) provided that if \( (x_n) \) is a sequence in X without a norm convergent subsequence and \( (x_n) \) converges to x with respect to \( \tau \) then
\[
\| x \| \leq \limsup_{n \to \infty} \| x_n \| \quad \text{for every sequence} \ (x_n) \ \text{in} \ X \ \text{such that} \ (x_n) \ \text{converges to} \ x.
\]
Definition 2.17 (Asymptotic Contraction)[8]: Let $\Phi$ denote the collection of all functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (1). $\phi$ is continuous.

(2). $\phi(t) < t, \forall t > 0$. Let $(M, d)$ be a complete metric space. A mapping $T : M \rightarrow M$ is said asymptotic contraction if for each integer $n \geq 1$, there exist a function $\phi_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $d(T^n(x), T^n(y)) \leq \phi_n d(x, y), x, y \in M$ and if $\phi_n \rightarrow \phi \in \Phi$ uniformly on the range of $d$.

Definition 2.18 :((P)$_\tau$-fixed point property)[4]: A mapping $T : C \rightarrow C$ is said to satisfy (P)$_\tau$ fixed point property if $T$ has a fixed point in every non empty convex $\tau$ sequentially closed subset $D$ of $C$ such that if $x \in D$, then each $\tau$-limit of a subsequence of $(T^n(x))$ belongs to $D$.

Theorem 2.19 [4]Let $X$ be an L-embedded Banach space. If a bounded sequence $(x_n)$ converges to 0 in abstract measure topology then
$$\limsup_{n \rightarrow \infty} \| x + x_n \|= \| x \| + \limsup_{n \rightarrow \infty} \| x_n \| \text{ for every } x \in X.$$

Theorem 2.20 [4] Let $\tau$ be a vector topology in an L-embedded Banach space $X,(x_n)$be a bounded sequence in $X$ such that the $\{x_n\}$ is relatively sequentially compact in an abstract measure topology and $r(x) = \limsup_{n \rightarrow \infty} \| x_n - x \|, \text{ where } x \in X$. Then,

(1) If the norm of $X$ is $\tau$-SLSC then the function $r$ is $\tau$-SLSC. (2) If $X$ has $KK(\tau)$ property and $(z_n)$ is a sequence in $X$ such that $(z_n)$ converges to $z$ with respect to $\tau$ and $(z_n)$ does not have a norm convergent subsequence then
$$r(z) < \limsup_{n \rightarrow \infty} r(z_n).$$

3 Main Results

Theorem 3.1 Let $X$ be an L-Embedded Banach Space with its abstract measure topology $\tau_\mu$, which is coarser than weak topology on $X$. Let $Y$ be a norm closed, reflexive L-Embedded subspaces of $X$. Then every non-expansive mapping $F : B_Y \rightarrow B_Y$ has a fixed point.

Proof: Let $z \in B_Y$. Define $F_n = (1 - \frac{1}{n})F + \frac{1}{n}z$. Then $F_n$ is a contraction. Hence $F_n$ has a unique fixed point, say, $x_n \in B_Y$. Now $(F(x_n))$ is a bounded sequence in $B_Y$. As $Y$ is reflexive, each bounded sequence contains a weakly convergent subsequence. Let $(F(x_{nk}))$ be the weak convergent subsequence of $(F(x_n))$ converging weakly to $y$. Then $(F(x_{nk})) \rightarrow y$ with respect to $\tau_\mu$. As $B_Y$ is
$\tau_\mu$ closed $y \in B_Y$. By the (2.5)($F(x_{nk})$)$\rightarrow y$ with respect to $\|\cdot\|$. As F is continuous ($F(F(x_{nk}))$)$\rightarrow F(y)$ with respect to $\|\cdot\|$. $x_{nk}$ is a fixed point of $F_{nk}$ gives $\lim(x_{nk}) = y$, which implies $y$ is a fixed point of $F$.

**Definition 3.2** ($\tau$-demi closed): Let $X$ and $Y$ be two Banach spaces and let $f$ be a mapping from $X$ into $Y$. Then $f$ is said to be $\tau$-demi closed if $(x_n)$ converges to $x$ with respect to $\tau$ and $(f(x_n))$ converges to $y$ with respect to norm imply $f(x) = y$.

**Theorem 3.3** Let $X$ be an L-embedded Banach space with abstract measure topology $\tau_\mu$. Let $Y$ be a norm closed subspace of $X$ and $B_Y$ a norm closed unit ball which is $\tau_\mu$ sequentially compact. If $F : B_Y \rightarrow B_Y$ is nonexpansive such that $I - F$ is demiclosed then $F$ has a fixed point.

**Proof:** By [2.7] there exists $(x_n) \in B_Y$ such that $\lim_{n \rightarrow \infty} \|(x_n) - F(x_n)\| = 0$. Hence $(I - F)(x_n)$ converges to 0. As $B_Y$ is $\tau_\mu$ sequentially compact, $(x_n)$ has a $\tau_\mu$ convergent subsequence, say $(x_{nk})$, which converges to some $x$ in $B_Y$. As $I - F$ is demiclosed $(I - F)(x) = 0$, which implies $x$ is a fixed point of $F$.

**Theorem 3.4** Let $X$ be an L-embedded Banach space endowed with abstract measure topology, which is coarser than weak topology. Let $Y$ be a norm closed L-embedded subspace of $X$. Let $F : B_Y \rightarrow B_Y$ be non-expansive, $\tau_\mu$ continuous mapping. Then $F$ has a fixed point.

**Proof:** Let $F_n = (1 - \frac{1}{n})F$. As $F_n$ is a contraction $F_n$ has a unique fixed point $x_n \in B_Y$. As $(F(x_n))$ is a bounded sequence there exists a subsequence $(F(x_{nk}))$ which either converges weakly to some $y$ in $Y$, or which converges to 0 with respect to $\tau_\mu$. As $\tau_\mu$ is weaker than weak topology, $(F(x_{nk}))$ converges to some $x$ in $Y$. Then $x_{nk} = F_n(x_{nk}) = (1 - \frac{1}{n})F(x_{nk})$. Taking limits with respect to $\tau_\mu$ topology and as $F$ is continuous $x$ is a fixed point of $F$.

**Definition 3.5**: A best approximation set $P_Y(x)$ is said to satisfy property $C^*$, if the chebyshev centre of $P_Y(x)$ is also a best approximation set for some $x^*$ in $X$.

**Definition 3.6**: A best approximation set $P_Y(x)$ is said to satisfy property $C$ with respect to a mapping $F : Y \rightarrow Y$, if $F(P_Y(x)) \subseteq P_Y(x)$ and $\overline{\text{co}}(F(P_Y(x)))$ is also a best approximation set for some $x^*$ in $X$. 


Theorem 3.7 Let $X$ be an $L$-embedded Banach space.$Y$ a proper norm closed $L$-embedded subspace of $X$, which has normal structure. Then any non expansive mapping $F : Y \to Y$ has a fixed point.

Proof: Let $\mu\{P_{Y}(x) : P_{Y}(x) \neq \phi, P_{Y}(x)$ is bounded, with the properties $C$* and $C$. Since elements of $\mu$ are weakly compact each descending chain in $\mu$ has a lower bound. Hence by Zorn’s lemma $\mu$ has a minimal element, say, $D_{0} = P_{Y}(x)$. Then $F(D_{0}) \subseteq D_{0}$, which implies $\overline{\phi}(F(D_{0})) \subseteq D_{0}$. Hence $F(\overline{\phi}(F(D_{0})) \subseteq \overline{\phi}(F(D_{0}))$, which implies $\overline{\phi}(F(D_{0})) \in \tau$. Hence $\overline{\phi}(F(D_{0})) = D_{0}$

Let $u \in c(D_{0})$. Then $r_{u}(D_{0}) = r(D_{0})$. Now $\| F_{u} - F_{v} \| \leq \| u - v \| \leq r(D_{0})$, $\forall v \in D_{0}$, implies $F(D_{0}) \in B(F(u), r(D_{0}))$. Hence, $D_{0} = \overline{\phi}(F(D_{0})) \subseteq B(F(u), r(D_{0}))$, $r_{F(u)}(D_{0}) = r(D_{0}) \Rightarrow F(u) \in C(D_{0})$. As $D_{0}$ is minimal element $C(D_{0}) = D_{0}$. Since $Y$ has normal structure $D_{0}$ must be a singleton set, which is fixed under $F$.

Theorem 3.8 Let $\tau$ be a vector topology coarser than norm topology in an $L$-embedded Banach space $X,(x_{n})$ be a bounded sequence in $X$ and let $f : X \to X$ be compact, homogeneous and onto, non-linear operator with respect to norm and also sequential continuous with respect to $\tau$. Define $r(x) = \limsup_{n \to \infty} \| f(x_{n}) - f(x) \|$, then

(1). Norm of $x$ is $\tau$-SLSC $\Rightarrow r$ is $\tau$-SLSC $\Rightarrow r$ is lower semi continuous with respect to norm.

(2). If $X$ has $KK(\tau)$ property and $(z_{n})$ is a sequence in $X$ such that $(z_{n})$ converges to $z$ with respect to $\tau$ and $(z_{n})$ does not have a norm convergent subsequence then $r(z) < \limsup_{n \to \infty} r(z_{n})$

Proof: $(x_{n})$ is bounded and $f : X \to X$ Compact, homogeneous operator, hence $(f(x_{n}))$ has a subsequence which converges in $X$. Let $f(x_{n_{k}}) \to f(y)$ and $(z_{n}) \to z$ in abstract measure topology. Then $r(z) \leq \limsup_{n \to \infty} r(z_{n})$. Hence $r$ is $\tau$-SLSC and lower semi continuous with respect norm. Proof 2 follows similarly.

Theorem 3.9 Let $X$ be an $L$-embedded Banach space and $\tau$ be a vector topology in $X$ coarser than the norm topology such that every $\tau$-sequentially compact subset of $X$ is $\tau$ compact and $X$ has $KK(\tau)$ property. Let a nonempty bounded convex and complete set $C \subseteq X$ be $\tau$-sequentially compact in an abstract measure topology. Then every asymptotic contractive mapping $T : C \to C$ has $(P)_{\tau}$ fixed point property.

Proof: $T$ is asymptotic contractive $\Rightarrow \limsup_{n \to \infty} \| T^{n}x - T^{n}y \| - \phi_{n} \| x - y \| \leq 0$ \limsup_{n \to \infty}, \limsup_{m \to \infty} \| T^{n}x - T^{m}y \| \leq \phi r(x_{y}) \leq r(x_{y}), \forall x, y \in C ; n > m -(1)$
Let $F$ be the family of all non empty convex $\tau$-sequentially closed subsets $K$, of $C$ such that if $y \in K$ and $z$ is a limit with respect to $\tau$ of a subsequence of $(T^ny)$, then $z \in K$. Let $D \in F$. By Zorn’s lemma we can find a $K_0 \in F$ which is minimal with respect to inclusion in the family $\{K \in F : K \subseteq D\}$. Let $x \in K_0$. We will show that the set $\{T_n(x)\}$ is relatively compact in the norm topology. Let $K_1$ be the set of all $z \in K_0$ at which the function $r_x$ attains its infimum on $K_0$. Then $K_1$ is non empty, $\tau$-sequentially closed and convex. Let $z \in K_1$ and $(n_k)$ be an increasing sequence such that $(T^{n_k}(z))$ converges to some $u$ with respect to $\tau$. Then $r_x(u) \leq \limsup_{k \to \infty} r_x(T^{n_k}(z)) \leq r_x(z)$. This shows that $u \in K_1$ and $K_1 \in F$. Consequently, $K_1 = K_0$ and in particular $r_x$ attains at $x$ its infimum on $K_0$. Suppose that there exists an increasing sequence $(n_k)$ such that $(T^{n_k}(x))$ does not have a norm convergent subsequence. We can assume that $(T^{n_k}(x))$ converges to some $u \in K_0$ with respect to $\tau$. Then $r_x(u) \leq \limsup_{k \to \infty} r_x(T^{n_k}(x)) \leq r_x(x)$ which is a contradiction. Now the result follows easily.

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