Majority Domination Edge Critical Graphs

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Abstract

This paper deals the graphs for which the removal of any edge changes the majority domination number of the graph. $\gamma_M$-critical edges, $\gamma_M$-redundant edges, $\gamma_M$-durable graphs and $\gamma_M$-critical graphs are studied. Further, majority domination critical edges and majority domination critical graphs are characterized.

Keywords: Majority Dominating Set, $\gamma_M$-critical edges and graphs

1 Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. Let $G = (V, E)$ be a finite graph and $v$ be a vertex in $V$. The closed neighborhood of $v$ is defined by $N[v] = N(v) \cup \{v\}$. The closed neighborhood of a set of vertices $S$ is denoted as $N[S]$ and is $\cup \lim_{s \in S} N[s]$.

Definition 1.1 A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a Majority Dominating Set if at least half of the vertices of $V$ are either in
S or adjacent to elements of $S$. (i.e.) $|N[S]| \geq \lceil \frac{|V(G)|}{2} \rceil$.

A majority dominating set $S$ is minimal if no proper subset of $S$ is a majority dominating set. The minimum cardinality of a minimal majority dominating set is called the majority domination number and is denoted by $\gamma_M(G)$. The maximum cardinality of a minimal majority dominating set is denoted by $\Gamma_M(G)$. Majority dominating sets have super hereditary property.

2 Change of Majority Domination in the case of removal of a single edge

The graphs in CER were characterised by Bauer et al. and H. B. Waliker and B. D. Acharya [5]. Let $(G - e)$ denotes the graph formed by removing an edge $e$ from $G$. Thus a graph $G$ has the property that $\gamma_M(G - e) = \gamma_M(G) + 1$, for all $e \in E(G)$. CER - Changing Edge Removal and UER - Unchanging Edge Removal.

Definition 2.1 (1). For any graph $G$, CER with respect to majority domination is defined by

CER$_M$ : $\gamma_M(G - e) \neq \gamma_M(G)$, for all $e \in E(G)$.

UER$_M$ : $\gamma_M(G - e) = \gamma_M(G)$, for all $e \in E(G)$.

(2). For any graph $G$, $E_M^0(G)$ and $E_M^+(G)$ are defined by

$E_M^0(G) = \{e \in E(G) : \gamma_M(G - e) = \gamma_M(G)\}$

$E_M^+(G) = \{e \in E(G) : \gamma_M(G - e) > \gamma_M(G)\}$.

Proposition 2.2 Let $G$ be a graph and $H$ be a spanning subgraph of $G$. Then $D_M(H) \subseteq D_M(G)$, $D_M(G)$ denotes the set of all majority dominating sets of $G$.

Corollary 2.3 Let $G$ be a graph and $x$ be any edge of $G$. Then $\gamma_M(G - x) \geq \gamma_M(G)$.

Proposition 2.4 Let $G$ be any graph and $x$ be any edge of $G$. Then exactly one of the following is true.

(i) $\gamma_M(G - x) = \gamma_M(G)$

(ii) $\gamma_M(G - x) = \gamma_M(G) + 1$
**Proof**: Let \( x = uv \) be any edge of \( G \) and \( (G - x) \) is a spanning subgraph of \( G \). By corollary 2.3, \( \gamma_M(G - x) \geq \gamma_M(G) \) ....................................(1)
\( \gamma_M(G - x) = \gamma_M(G) + k \) for some \( k \geq 0 \) ........................(2)
**Claim**: \( k \leq 1 \). Let \( H = (G - x) \). Let \( D \) be any minimum majority dominating set of \( G \).

Case (i): \( u \) and \( v \) are adjacent in \( D \).

Case (ii): \( u \) and \( v \) are adjacent in \( V - D \). (Similar proof if \( u \in V - D \) and \( v \in D \).

Then \( D \cup \{ v \} \) is a majority dominating set of \( (G - x) \). \( \gamma_M(G - x) \leq |D \cup \{ v \}| = |D| + 1 \). \( \gamma_M(G - x) \leq \gamma_M(G) + 1 \). By (2), \( \gamma_M(G) + k \leq \gamma_M(G) + 1 \) implies \( k \leq 1 \). Hence the theorem. 

**Definition 2.5** Let \( G = (V, E) \) be any graph and \( x \) be any edge of \( G \).

(1). An edge \( x \) is \( \gamma_M \)-critical if \( \gamma_M(G - x) = \gamma_M(G) + 1 \). (2). An edge \( x \) of \( G \) is called \( \gamma_M \)-redundant if \( \gamma_M(G - x) = \gamma_M(G) \). (3). A graph \( G \) is called \( \gamma_M \)-durable if each edge of \( G \) is \( \gamma_M \)-redundant.

### 3 Characterisation of \( \gamma_M \)-critical edge

**Theorem 3.1** An edge \( x = uv \) of a graph \( G \) is \( \gamma_M \)-critical if and only if for every minimum majority dominating set \( D \), the following three conditions hold:

(i) \( u \) and \( v \) do not both belong to \( D \) or both belong to \( V - D \).

(ii) If \( v \in V - D \) then \( N(v) \cap D = \{ u \} \). A similar result holds if \( u \in V - D \).

(iii) Suppose \( u \in D \) and \( v \in V - D \). Then \( |N[D] - \{ v \}| < \left\lceil \frac{p}{2} \right\rceil \).

**Proof**: Let \( x = uv \) be a \( \gamma_M \)-critical edge. Let \( D \) be a minimum majority dominating set of \( G \). Then \( |N_G[D]| \geq \left\lceil \frac{p}{2} \right\rceil \).

If the condition (i) is not true for a particular \( D \in D_{M_0}(G) \), then either \( u, v \in D \) or \( u, v \in V - D \). In either case, \( D \) is again a majority dominating set of \( (G - x) \). Therefore, \( \gamma_M(G - x) \leq |D| = \gamma_M(G) \).

By corollary 2.3, \( \gamma_M(G - x) \geq \gamma_M(G) \). Hence \( \gamma_M(G - x) = \gamma_M(G) \) and it follows that \( x \) is a \( \gamma_M \)-redundant edge, a contradiction.

Suppose condition (ii) is not true. Then \( D \) is a majority dominating set of \( (G - x) \), a contradiction.

Suppose condition (iii) is not true. Then \( u \in D \), \( v \in V - D \) and \( |N_G[D] - \{ v \}| \geq \left\lceil \frac{p}{2} \right\rceil \).
Claim: Suppose \( \gamma(G - x) = \gamma(G) + 1 \).

Suppose this is not true, then by proposition 2.4, \( \gamma(G - x) = \gamma(G) \). Hence \( |N[v] \cap D| \geq 2 \) are \( |N[D] - \{v\}| \geq \left\lceil \frac{p}{2} \right\rceil \). Therefore condition (ii) or (iii) is not satisfied which is a contradiction. Thus \( x \) is a \( \gamma_M \)-critical edge.

\section{\( \gamma_M \)-critical Graph}

\textbf{Definition 4.1} A graph \( G \) is called \( \gamma_M \)-critical if for every edge \( x \) of \( G \),
\[ \gamma_M(G - x) = \gamma_M(G) + 1. \]

\textbf{Characterisation of \( \gamma_M \)-critical Graph}

\textbf{Theorem 4.2} A graph \( G \) is \( \gamma_M \)-critical if and only if
\[ G = (K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_s}) \cup [p - (r_1 + r_2 + \ldots + r_s + s)]K_1 \]
where \( 2 \leq (r_1 + r_2 + \ldots + r_s + s) \leq \left\lceil \frac{p}{2} \right\rceil \).

\textbf{Proof:} Let \( D \) be a \( \gamma_M \)-set of \( G \). Suppose \( D \) is not independent. Then there is an edge \( x = uv \in E(G) \) for some \( u, v \in D \). Then \( \gamma_M(G - x) = |D| = \gamma_M(G) \), a contradiction to \( G \) is \( \gamma_M \)-critical. Therefore \( D \) is independent.

Suppose \( V - D \) is not independent, then there is an edge \( x = uv \in E(G) \) for some \( u, v \in V - D \). Then \( \gamma_M(G - x) = |D| = \gamma_M(G) \), a contradiction. Therefore \( V - D \) is independent.

Suppose \( d(u) \geq 2 \) for some \( u \in V - D \). Since \( V - D \) is independent, there exist two vertices \( v, w \) in \( D \) such that \( x = uv \) and \( y = uw \in E(G) \). Then \( \gamma_M(G - x) = \gamma_M(G) \), a contradiction. Hence, \( d(u) \leq 1 \) for all \( u \in V - D \).

Thus \( G = (K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_s}) \cup [p - (r_1 + r_2 + \ldots + r_s + s)]K_1 \).

\textbf{Claim:} \( 2 \leq (r_1 + r_2 + \ldots + r_s + s) \leq \left\lceil \frac{p}{2} \right\rceil \).

Suppose \( (r_1 + \ldots + r_s + s) > \left\lceil \frac{p}{2} \right\rceil \), then \( (r_1 + r_2 + \ldots + r_s + s) \geq \left\lceil \frac{p}{2} \right\rceil + 1 \). Let \( D \) be a \( \gamma_M \)-set of \( G \). Let \( x = uv \) be any edge of \( G \) such that \( u \in D \) and \( v \in V - D \).
Then $D$ is also a $\gamma_M$-set of $(G-x)$ since $(r_1 + r_2 + \ldots + r(i-1) + r_s + s) \geq \lceil \frac{p}{2} \rceil$. Therefore $\gamma_M(G-x) = \gamma_M(G)$, a contradiction. Hence $2 \leq (r_1 + r_2 + \ldots + r_s + s) \leq \lceil \frac{p}{2} \rceil$.

Therefore $G = (K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_s}) \cup [p - (r_1 + r_2 + \ldots + r_s + s)]K_1$, where $2 \leq (r_1 + r_2 + \ldots + r_s + s) \leq \lceil \frac{p}{2} \rceil$.

Conversely, let $G = (K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_s}) \cup [p - (r_1 + r_2 + \ldots + r_s + s)]K_1$ where $2 \leq (r_1 + r_2 + \ldots + r_s + s) \leq \lceil \frac{p}{2} \rceil$.

(i.e.), $G = (K_{1,r_1} \cup K_{1,r_2} \cup \ldots \cup K_{1,r_s}) \cup \lambda K_1$ where $\lambda = p - (r_1 + r_2 + \ldots + r_s + s), \ 2 \leq p - \lambda \leq \lceil \frac{p}{2} \rceil$.

In the above graph $G$, let $V(K_{1,r_i}) = \{v_i, v_{i1}, v_{i2}, \ldots, v_{ir_i}\}, \ 1 \leq i \leq s$ and $V(\lambda K_1) = \{x_1, x_2, \ldots, x_\lambda\}$. Now $D = \{v_i, v_{i1}, v_{i2}, \ldots, v_s, x_1, x_2, \ldots, x_t\}$ is a $\gamma_M$-set of $G$ where $t = \lambda - \lceil \frac{\lambda}{2} \rceil$. In general, any $\gamma_M$-set of $G$ contains all $v_i$’s and $t$ vertices from $\{x_1, x_2, \ldots, x_\lambda\}$. Let $x \in E(G)$ and $x = v_iv_{ij}, \ 1 \leq i \leq s, \ 1 \leq j \leq r_i$ in $G-x$, where $v_{ij}$ is an isolate. Moreover $D$ is not a $\gamma_M$-set of $G-x$. But $D \cup \{v_{ij}\}$ is a $\gamma_M$-set of $G-x$. Therefore $\gamma_M(G-x) = \gamma_M(G)+1$ for all $x \in E(G)$. Hence $G$ is a $\gamma_M$-critical graph.

**Corollary 4.3** A graph $G$ is $\gamma_M$-critical if and only if for every edge $x=uv$ in $G$ and for every minimum majority dominating set $D$ in $G$, the following three conditions hold:

(i) $u \in D$ and $v \in V-D$ or vice versa.

(ii) If $v \in V-D$ then $N(v) \cap D = \{u\}$ and if $u \in V-D$ then $N(u) \cap D = \{v\}$.

(iii) Suppose $v \in V-D$ then $|N[D] - \{v\}| < \lceil \frac{\lambda}{2} \rceil$.

**Theorem 4.4** Let $G$ be a $\gamma_M$-critical graph. Then $G$ is not connected.

**Theorem 4.5** For any graph $G$, the following statements are equivalent:

(a) $G$ is $\gamma_M(G)$-critical.
(b) $G = (K_{1,r_1} \cup K_{1,r_2} \cup ... \cup K_{1,r_s}) \cup [p - (r_1 + ... + r_s + s)]K_1$
where $2 \leq (r_1 + ... + r_s + s) \leq \lceil \frac{p}{2} \rceil$.

(c) $\gamma_M(G) = \lceil \frac{p - 2q}{2} \rceil$.

References


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