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On the Primes in the Interval $[3n, 4n]$

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Abstract

For the old question whether there is always a prime in the interval $[kn, (k+1)n]$ or not, the famous Bertrand's postulate gave an affirmative answer for $k = 1$. It was first proved by P.L. Chebyshev in 1850, and an elegant elementary proof was given by P. Erdős in 1932 (reproduced in [2, pp. 171-173]). M. El Bachraoui used elementary techniques to prove the case $k = 2$ in 2006 [1]. This paper gives a proof of the case $k = 3$, again without using the prime number theorem or any deep analytic result. In addition we give a lower bound for the number of primes in the interval $[3n, 4n]$, which shows that as n tends to infinity, the number of primes in the interval $[3n, 4n]$ goes to infinity.

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0 Notations

Throughout this paper, we let n run through the positive integers and p run through the primes. We also let $\pi(n)$ be the prime counting function, which counts the number of primes not exceeding n. Further define

$$f(x) = \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}}$$

and

$$g(x) = \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}}.$$

1 Lemmas

Lemma 1.1. *If $n \geq 8$, then*

$$\pi(n) \leq \frac{n}{2}.$$

Proof. This is trivial since $1, 9$ and all even positive integers are not prime.

Lemma 1.2. *If x is a positive real number, then*

$$\prod_{p \le x} p \le 4^x.$$

Proof. See [2, pp. 167-168].

Lemma 1.3. *We have*

$$g(n) < n! < f(n)$$

Proof. See [3].

Lemma 1.4. *For a fixed constant $c \ge \frac{1}{12}$, define the function*

$$h_1(x) = \frac{f(x+c)}{g(c)g(x)}.$$

Then for $x \ge \frac{1}{2}$, $h_1(x)$ is increasing.

Proof. It suffices to prove that the function

$$H_1(x) = (x+c)^{x+c+\frac{1}{2}} x^{-x-\frac{1}{2}} e^{\frac{1}{12(x+c)} - \frac{1}{12x+1}}$$

is increasing for $x > \frac{1}{2}$. Indeed, we have

$$H_1'(x) = H_1(x)\left(\left(\frac{1}{2(x+c)} - \frac{1}{12(x+c)^2} + \ln(x+c)\right) - \left(\frac{1}{2x} - \frac{1}{12\left(x+\frac{1}{12}\right)^2} + \ln x\right)\right),$$

where $H_1(x) > 0$. Let

$$F_1(x) = \frac{1}{2x} + \ln x.$$

As $-\frac{1}{12(x+c)^2} \ge \frac{1}{12\left(x+\frac{1}{12}\right)^2}$, it suffices to prove that $F_1(x)$ is increasing, so that

$$\left(\frac{1}{2(x+c)} + \ln(x+c)\right) - \left(\frac{1}{2x} + \ln x\right) \ge 0.$$

We actually have

$$F_1'(x) = -\frac{1}{2x^2} + \frac{1}{x} = \frac{2x-1}{2x^2},$$

which must be non-negative for all $x \ge \frac{1}{2}$. Therefore, the desired result follows.

Lemma 1.5. *For a fixed positive constant* c*, define the function*

$$h_2(x) = \frac{f(c)}{g(x)g(c-x)}.$$

Then when $\frac{1}{2} \le x < \frac{c}{2}$, $h_2'(x) > 0$*; when* $x = \frac{c}{2}$, $h_2'(x) = 0$ *and when* $\frac{c}{2} < x \le c - \frac{1}{2}$, $h_2'(x) < 0$.

Proof. It suffices to prove that the function

$$H_2(x) = x^{x+\frac{1}{2}}(c-x)^{c-x+\frac{1}{2}}e^{\frac{1}{12x+1}+\frac{1}{12(c-x)+1}}$$

has the following property: when $\frac{1}{2} \le x < \frac{c}{2}$, $H_2'(x) < 0$; when $x = \frac{c}{2}$, $H_2'(x) = 0$ and when $\frac{c}{2} < x \le c - \frac{1}{2}$, $H_2'(x) > 0$. Indeed, we have

$$H_2'(x) = x^{x+\frac{1}{2}}(c-x)^{c-x+\frac{1}{2}}e^{\frac{1}{12x+1}-\frac{1}{12(c-x)+1}}(F_2(x) - F_2(c-x)),$$

where

$$F_2(x) = \frac{12}{(12(c-x)+1)^2} + \frac{1}{2x} + \ln x.$$

Clearly, for $\frac{1}{2} \le x < c$,

$$x^{x+\frac{1}{2}}(c-x)^{c-x+\frac{1}{2}}e^{\frac{1}{12x+1}-\frac{1}{12(c-x)+1}} > 0.$$

Next, we actually have

$$F_2'(x) = \frac{288}{(12(c-x)+1)^3} + \frac{2x-1}{2x^2},$$

which must be positive for all $\frac{1}{2} \le x < c$. Thus whenever $\frac{1}{2} \le x \le c - \frac{1}{2}$, $F_2(x)$ is increasing while $F_2(c-x)$ is decreasing, implying that there is at most one value of x with $\frac{1}{2} \le x \le c - \frac{1}{2}$ satisfying $F_2(x) = F_2(c-x)$. It is clear that $x = \frac{c}{2}$ is such a value. It follows that when $\frac{1}{2} \le x < \frac{c}{2}$, $H_2'(x) < 0$ and when $\frac{c}{2} < x \le c - \frac{1}{2}$, $H_2'(x) > 0$.

2 Main Results

Now, suppose $n > e^{12}$. The product of all primes $p \in (3n, 4n]$, if any, must divide $\binom{4n}{3n}$. Let $\beta(p)$ be the power of p in the prime factorization of $\binom{4n}{3n}$. Let

$$\binom{4n}{3n} = T_1 T_2 T_3$$

where

$$T_1 = \prod_{p \leq \sqrt{4n}} p^{\beta(p)}, \quad T_2 = \prod_{\sqrt{4n} < p \leq 3n} p^{\beta(p)} \quad \text{and} \quad T_3 = \prod_{3n < p \leq 4n} p^{\beta(p)}.$$

Bounding each multiplicand in T_1 from above by $4n$ (see [3, p. 24]) and applying Lemma 1.1,

$$T_1 < (4n)^{\pi(\sqrt{4n})} \leq (4n)^{\frac{\sqrt{4n}}{2}} = (4n)^{\sqrt{n}}.$$

Consider T_2. As the prime factorization of $\binom{n}{j}$ in [3, p. 24] manifests, for $\sqrt{4n} < p \leq 3n$, $\beta(p) \leq 1$.

Let $x > 0$ and let $[x]$ be the greatest integer less than or equal to x. Define $\{x\} = x - [x]$. Let r and s be real numbers satisfying $s > r \geq 1$. Observe that number of integers in the interval $(s - r, s]$ is $[s] - [s - r]$, which is $[r]$ if $\{s\} \geq \{r\}$ and $[r] + 1$ if $\{s\} < \{r\}$. Let N be the set of all positive integers. We define

$$\begin{Bmatrix} s \\ r \end{Bmatrix} = \frac{\displaystyle\prod_{k \in (s-r, s] \cap N} k}{\displaystyle\prod_{k \in (0, r] \cap N} k} = \delta(r, s) \binom{[s]}{[r]},$$

where $\delta(r, s) = 1$ if $\{s\} \geq \{r\}$ and $\delta(r, s) = [s - r] + 1$ if $\{s\} < \{r\}$. In both cases, $\delta(r, s) \leq s$.

Now let $A = \begin{Bmatrix} 4n/3 \\ n \end{Bmatrix}$, $B = \begin{Bmatrix} 2n \\ 3n/2 \end{Bmatrix}$, $C = \begin{Bmatrix} 4n/17 \\ 3n/13 \end{Bmatrix}$ and $D = \begin{Bmatrix} 2n/7 \\ 4n/15 \end{Bmatrix}$.

We have the following observations:

- $\displaystyle\prod_{\sqrt{4n} < p \leq \frac{n}{6}} p \leq \prod_{p \leq \frac{n}{6}} p \leq 4^{\frac{n}{6}}$ (by Lemma 1.2)

- If $\dfrac{n}{6} < p \leq \dfrac{2n}{11}$, then

$$2p < \frac{n}{2} < 3p < 8p < \frac{3n}{2} < 9p < 11p \leq 2n.$$

  Hence $\displaystyle\prod_{\frac{n}{6} < p \leq \frac{2n}{11}} p$ divides B.

- If $\frac{2n}{11} < p \leq \frac{4n}{21}$, then

$$p < \frac{n}{3} < 2p < 5p < n < 6p < 7p \leq \frac{4n}{3}.$$

Hence $\prod_{\frac{2n}{11} < p \leq \frac{4n}{21}} p$ divides A.

- If $\frac{4n}{21} < p \leq \frac{n}{5}$, then

$$5p \leq n < 6p < 15p \leq 3n < 16p < 20p \leq 4n < 21p.$$

Hence $\beta(p) = 0$.

- If $\frac{n}{5} < p \leq \frac{2n}{9}$, then

$$p < \frac{n}{3} < 2p < 4p < n < 5p < 6p \leq \frac{4n}{3}.$$

Hence $\prod_{\frac{n}{5} < p \leq \frac{2n}{9}} p$ divides A.

- If $\frac{2n}{9} < p \leq \frac{3n}{13}$, then

$$4p < n < 5p < 13p < 3n < 14p < 17p < 4n < 18p.$$

Hence $\beta(p) = 0$.

- $\prod_{\frac{3n}{13} < p \leq \frac{4n}{17}} p$ divides C.

- If $\frac{4n}{17} < p \leq \frac{n}{4}$, then

$$4p \leq n < 5p < 12p \leq 3n < 13p < 16p \leq 4n < 17p.$$

Hence $\beta(p) = 0$.

- If $\frac{n}{4} < p \leq \frac{4n}{15}$, then

$$p < \frac{n}{3} < 2p < 3p < n < 4p < 5p \leq \frac{4n}{3}.$$

Hence $\prod_{\frac{n}{4} < p \leq \frac{4n}{15}} p$ divides A.

- $\displaystyle\prod_{\frac{4n}{15}<p\leq\frac{2n}{7}} p$ divides D.
- If $\dfrac{2n}{7} < p \leq \dfrac{3n}{10}$, then
$$3p < n < 4p < 10p \leq 3n < 11p < 13p < 4n < 14p.$$
Hence $\beta(p) = 0$.
- If $\dfrac{3n}{10} < p \leq \dfrac{n}{3}$, then
$$p < \frac{n}{2} < 2p < 4p < \frac{3n}{2} < 5p < 6p \leq 2n.$$
Hence $\displaystyle\prod_{\frac{3n}{10}<p\leq\frac{n}{3}} p$ divides B.
- If $\dfrac{n}{3} < p \leq \dfrac{4n}{9}$, then
$$\frac{n}{3} < p < 2p < n < 3p \leq \frac{4n}{3}.$$
Hence $\displaystyle\prod_{\frac{n}{3}<p\leq\frac{4n}{9}} p$ divides A.
- If $\dfrac{4n}{9} < p \leq \dfrac{n}{2}$, then
$$2p \leq n < 3p < 6p \leq 3n < 7p < 8p \leq 4n < 9p.$$
Hence $\beta(p) = 0$.
- If $\dfrac{n}{2} < p \leq \dfrac{2n}{3}$, then
$$\frac{n}{2} < p < 2p < \frac{3n}{2} < 3p \leq 2n.$$
Hence $\displaystyle\prod_{\frac{n}{2}<p\leq\frac{2n}{3}} p$ divides B.
- If $\dfrac{2n}{3} < p \leq \dfrac{3n}{4}$, then
$$p < n < 2p < 4p \leq 3n < 5p < 4n < 6p.$$
Hence $\beta(p) = 0$.

- If $\frac{3n}{4} < p \le \frac{4n}{5}$, then

$$\frac{n}{2} < p < \frac{3n}{2} < 2p \le 2n.$$

  Hence $\prod_{\frac{3n}{4} < p \le \frac{4n}{5}} p$ divides B.

- If $\frac{4n}{5} < p \le n$, then

$$p \le n < 2p < 3p \le 3n < 4p \le 4n < 5p.$$

  Hence $\beta(p) = 0$.

- $\prod_{n < p \le \frac{4n}{3}} p$ divides A.

- If $\frac{4n}{3} < p \le \frac{3n}{2}$, then

$$n < p < 2p \le 3n < 4n < 3p.$$

  Hence $\beta(p) = 0$.

- $\prod_{\frac{3n}{2} < p \le 2n} p$ divides B.

- If $2n < p \le 3n$, then

$$n < p \le 3n < 4n < 2p.$$

  Hence $\beta(p) = 0$.

Therefore, to summarize, we get

$$T_2 \le 4^{\frac{n}{6}} ABCD.$$

Note that by Lemma 1.3,

$$\begin{aligned}\binom{4n}{3n} &= \frac{(4n)!}{(3n)!n!} \\ &> \frac{g(4n)}{f(3n)f(n)} \\ &= \frac{2}{\sqrt{6\pi n}} e^{\frac{1}{48n+1} - \frac{1}{36n} - \frac{1}{12n}} \left(\frac{256}{27}\right)^n,\end{aligned}$$

and similarly,

$$A = \left\{ {4n/3 \atop n} \right\} \le \frac{4n}{3} \binom{[4n/3]}{n} = \frac{4n}{3} \cdot \frac{\left[\frac{4n}{3}\right]!}{n!\left(\left[\frac{4n}{3}\right] - n\right)!}$$

$$< \frac{4n}{3} \cdot \frac{f\left(\left[\frac{4n}{3}\right]\right)}{g(n)g\left(\left[\frac{4n}{3}\right] - n\right)}$$

$$\le \frac{4n}{3} \cdot \frac{f\left(\frac{4n}{3}\right)}{g(n)g\left(\frac{n}{3}\right)} \quad \text{(by Lemma 1.4)}$$

$$= \frac{4n}{3}\sqrt{\frac{2}{\pi n}} e^{\frac{1}{16n} - \frac{1}{12n+1} - \frac{1}{4n+1}} \left(\frac{4^{\frac{4}{3}}}{3}\right)^n,$$

$$B = \left\{ {2n \atop 3n/2} \right\} \le 2n \binom{2n}{[3n/2]} = 2n \cdot \frac{\left[\frac{3n}{2}\right] + 1}{2n - \left[\frac{3n}{2}\right]} \binom{2n}{[3n/2] + 1}$$

$$< 2n \cdot \frac{\frac{3n}{2} + 1}{2n - \frac{3n}{2}} \cdot \frac{f(2n)}{g\left(\left[\frac{3n}{2}\right] + 1\right) g\left(2n - \left(\left[\frac{3n}{2}\right] + 1\right)\right)}$$

$$< (6n + 4) \cdot \frac{f(2n)}{g\left(\frac{3n}{2}\right) g\left(2n - \frac{3n}{2}\right)} \quad \text{(by Lemma 1.5)}$$

$$= \frac{12n + 8}{\sqrt{3\pi n}} e^{\frac{1}{24n} - \frac{1}{18n+1} - \frac{1}{6n+1}} \left(\frac{16}{3^{\frac{3}{2}}}\right)^n,$$

$$C = \left\{ {4n/17 \atop 3n/13} \right\} \le \frac{4n}{17} \binom{[4n/17]}{[3n/13]} = \frac{4n}{17} \cdot \frac{\left[\frac{3n}{13}\right] + 1}{\left[\frac{4n}{17}\right] - \left[\frac{3n}{13}\right]} \binom{[4n/17]}{[3n/13] + 1}$$

$$\le \frac{4n}{17} \cdot \frac{\frac{3n}{13} + 1}{\frac{4n}{17} - 1 - \frac{3n}{13}} \cdot \frac{f\left(\left[\frac{4n}{17}\right]\right)}{g\left(\left[\frac{3n}{13}\right] + 1\right) g\left(\left[\frac{4n}{17}\right] - \left(\left[\frac{3n}{13}\right] + 1\right)\right)}$$

$$< \frac{4n}{17}\cdot\frac{51n+221}{n-221}\frac{f\left(\frac{4n}{17}\right)}{g\left(\frac{3n}{13}\right)g\left(\frac{4n}{17}-\frac{3n}{13}\right)} \quad \text{(by Lemmas 1.4 and 1.5)}$$
$$= \frac{4n}{17}\cdot\frac{51n+221}{n-221}\cdot\frac{26}{\sqrt{6\pi n}}e^{\frac{17}{48n}-\frac{13}{36n+13}-\frac{221}{12n+221}}\left(221^{\frac{1}{221}}\left(\frac{13}{3}\right)^{\frac{3}{13}}\left(\frac{4}{17}\right)^{\frac{4}{17}}\right)^{n},$$

and

$$D = \left\{\begin{matrix}2n/7\\4n/15\end{matrix}\right\} \le \frac{2n}{7}\binom{[2n/7]}{[4n/15]} = \frac{2n}{7}\cdot\frac{\left[\frac{4n}{15}\right]+1}{\left[\frac{2n}{7}\right]-\left[\frac{4n}{15}\right]}\binom{[2n/7]}{[4n/15]+1}$$
$$\le \frac{2n}{7}\cdot\frac{\frac{4n}{15}+1}{\frac{2n}{7}-1-\frac{4n}{15}}\cdot\frac{f\left(\left[\frac{2n}{7}\right]\right)}{g\left(\left[\frac{4n}{15}\right]+1\right)g\left(\left[\frac{2n}{7}\right]-\left(\left[\frac{4n}{15}\right]+1\right)\right)}$$
$$< \frac{2n}{7}\cdot\frac{28n+105}{2n-105}\cdot\frac{f\left(\frac{2n}{7}\right)}{g\left(\frac{4n}{15}\right)g\left(\frac{2n}{7}-\frac{4n}{15}\right)} \quad \text{(by Lemmas 1.4 and 1.5)}$$
$$= \frac{4n^2+15n}{2n-105}\cdot\frac{15}{\sqrt{2\pi n}}e^{\frac{7}{24n}-\frac{5}{16n+5}-\frac{35}{8n+35}}\left(\left(\frac{105}{2}\right)^{\frac{2}{105}}\left(\frac{15}{4}\right)^{\frac{4}{15}}\left(\frac{2}{7}\right)^{\frac{2}{7}}\right)^{n}.$$

Therefore

$$T_3 = \binom{4n}{3n}\frac{1}{T_1T_2} > \binom{4n}{3n}\frac{1}{(4n)^{\sqrt{n}}4^{\frac{n}{6}}ABCD}$$
$$> \frac{\sqrt{3}\pi^{\frac{3}{2}}}{4160}e^{E}M^{n}(4n)^{-\sqrt{n}}\cdot\frac{n^{-\frac{3}{2}}(n-221)(2n-105)}{(3n+2)(3n+13)(4n+15)}$$
$$> \frac{\sqrt{3}\pi^{\frac{3}{2}}}{4160}e^{E}M^{n}(4n)^{-\sqrt{n}}\cdot\frac{n^{-\frac{3}{2}}n^{2}}{(4n)(4n)(5n)}$$
$$= \frac{\sqrt{3}\pi^{\frac{3}{2}}}{332800}e^{E}M^{n}(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}$$

where

$$E = \frac{1}{48n+1} - \frac{1}{36n} - \frac{1}{12n} - \frac{1}{16n} + \frac{1}{12n+1} + \frac{1}{4n+1} - \frac{1}{24n} + \frac{1}{18n+1} + \frac{1}{6n+1} - \frac{17}{48n} + \frac{13}{36n+13} + \frac{221}{12n+221} - \frac{7}{24n} + \frac{5}{16n+5} + \frac{35}{8n+35}$$

and

$$M = \frac{256}{7}\left(\frac{1}{4}\right)^{\frac{4}{3}}(3)\frac{3^{\frac{3}{2}}}{16}\left(\frac{1}{221}\right)^{\frac{1}{221}}\left(\frac{3}{13}\right)^{\frac{3}{13}}\left(\frac{17}{4}\right)^{\frac{4}{17}}\left(\frac{2}{105}\right)^{\frac{2}{105}}\left(\frac{4}{15}\right)^{\frac{4}{15}}\left(\frac{7}{2}\right)^{\frac{2}{7}}4^{-\frac{1}{6}} > 1.$$

Obviously

$$\lim_{n\to\infty} e^E = 1.$$

Moreover, we have

$$\ln\left(M^n(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}\right) = n\ln M - \sqrt{n}\ln(4n) - \frac{5}{2}\ln n.$$

When n tends to infinity, it is easy to check that $\sqrt{n}\ln(4n) = o(n)$ and $\ln n = o(n)$. Thus, $\ln\left(M^n(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}\right)$ goes to infinity and so does $M^n(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}$.

It follows that

$$\lim_{n\to\infty} T_3 = +\infty,$$

which means that there exists some n_0 such that for all $n \geq n_0$, $T_3 > 1$. In fact, it is routine to check (using WolframAlpha for instance) that when $n > e^{12}$, $\frac{\sqrt{3}\pi^{\frac{3}{2}}}{332800}e^E M^n(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}$ is always greater than 1 and so $T_3 > 1$. Direct verification, on the other hand, ensures that there is always a prime in the interval $[3n, 4n]$ for all positive integers $n < e^{12}$. Therefore, our desired result ensues:

Theorem 2.1. *For every positive integer n, there is a prime in the interval $[3n, 4n]$. Plainly, it follows that when $n \geq 2$, there is always a prime in the interval $(3n, 4n)$.*

Corollary 2.2. *If $n \geq 3$, then there is a prime in the interval $\left(n, \frac{4(n+2)}{3}\right)$.*

Proof. If $n \equiv 0 \pmod 3$, then the result follows directly from Theorem 2.1. If $n \equiv 1 \pmod 3$, then by Theorem 2.1 there exists a prime $p \in \left(n+2, \frac{4(n+2)}{3}\right)$. If $n \equiv 2 \pmod 3$, then by Theorem 2.1 there exists a prime $p \in \left(n+1, \frac{4(n+1)}{3}\right)$.

Next, we establish a lower bound for the number of primes in the interval $[3n, 4n]$. Bounding each prime in the interval from above by $4n$, we have the following

Theorem 2.3. *For* $n \geq 4$*, the number of primes in the interval* $(3n, 4n)$ *is at least*

$$\log_{4n}\left(\frac{\sqrt{3}\pi^{\frac{3}{2}}}{332800}e^{E}M^{n}(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}\right).$$

Note that

$$\begin{aligned}
&\log_{4n}\left(\frac{\sqrt{3}\pi^{\frac{3}{2}}}{332800}e^{E}M^{n}(4n)^{-\sqrt{n}}n^{-\frac{5}{2}}\right) \\
&= \frac{-\frac{5}{2}\ln n + n\ln M - \sqrt{n}\ln(4n) + E + \ln\left(\frac{\sqrt{3}\pi^{\frac{3}{2}}}{332800}\right)}{\ln n + \ln 4} \\
&= \frac{n\ln M - \sqrt{n}\ln(4n) + E + \ln\left(\frac{\sqrt{3}\pi^{\frac{3}{2}}}{332800}\right) + \frac{5}{2}\ln 4}{\ln n + \ln 4} - \frac{5}{2} \\
&> \frac{n\left(\ln M - \frac{\ln(4n)}{\sqrt{n}}\right)}{2\ln n} - \frac{5}{2}.
\end{aligned}$$

Now check that $\lim_{n\to\infty}\frac{\ln(4n)}{\sqrt{n}} = 0$. Moreover, it is obvious that

$$\lim_{n\to\infty}\frac{n}{\ln n} = +\infty.$$

Thus we have the following

Theorem 2.4. *As* n *tends to infinity, the number of primes in the interval* $[3n, 4n]$ *goes to infinity. In other words, for every positive integer* m*, there exists a positive integer* L *such that for all* $n \geq L$*, there are at least* m *primes in the interval* $[3n, 4n]$*.*

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