A Classification of Surfaces of Revolution
in Lorentz-Minkowski Space

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Abstract. In this study, the surfaces of revolution whose axes of revolution are spacelike and timelike are classified under the condition \( \tilde{\Gamma}^{11}_1 \Psi_i = \lambda_i \Psi_i \), \( \lambda_i \in \mathbb{R} \), \( (i = 1, 2, 3) \), where \( \tilde{\Gamma}^{11}_1 \) is one of the Christoffel-like operators with non-degenerate metric in the 3-dimensional Lorentz-Minkowski space.

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1. Introduction

The geometry of surfaces of revolution has been studied widely in Euclidean space \( E^3 \) as well as Lorentz-Minkowski space \( E^3_1 \). It is well known that induced metric on a surface \( M \) in \( E^3_1 \) can be non-degenerate or degenerate. If the induced metric is non-degenerate, then \( M \) is called a semi-Riemannian surface.

Relative to Takahashi’s theorems [12] for minimal submanifolds, the idea of submanifolds of finite type in Euclidean space was introduced by Chen [3]. As a generalization of Takahashi’s theorem for the case of hypersurfaces, Garay [6] considered the hypersurfaces satisfying the condition \( \Delta x = Ax \), where \( x \) is an isometric immersion from \( M \) to \( \mathbb{R}^{n+1} \), \( \Delta \) is the Laplacian on \( M \) and \( A \) is a \((n+1)\)-dimensional diagonal matrix and the theory is recently greatly developed. The surfaces of revolution and ruled surfaces in Euclidean 3-space are studied in [4] and [1], respectively, such that its Gauss map \( \xi \) satisfies the condition \( \Delta \xi = A \xi \), \( A \in Mat(3, \mathbb{R}) \).
O.J. Garay [7] investigated the surfaces of revolution, in $E^3$ Euclidean space, whose component functions are eigenfunctions of its Laplacian and he saw that these surfaces must be a Catenoid, a sphere or a right circular cylinder.

Also, the Lorentz version of the non-degenerate surfaces $M^2_s$, with index $s = 0, 1$ in $\mathbb{R}^3_1$, are classified under the condition $\Delta H = \lambda H$ by Ferrandez and Lucas in [5]. They proved that $M^2_s$ is a zero mean curvature surface everywhere, either an open piece of a B-scroll surface or an open piece of the surfaces $S^1(r) \times \mathbb{R}$, $H^1(r) \times \mathbb{R}$, $S^1_1(r) \times \mathbb{R}$, $H^2(r)$, $S^1_1(r)$.

In [8], Kaimakamis and Papantoniou classified the surfaces of revolution without parabolic points, in the 3-dimensional Lorentz-Minkowski space, under the condition

$$\Delta^H \vec{r} = A \vec{r},$$

where $\Delta^H$ is the Laplace operator with respect to the second fundamental form and $A$ is a real $3 \times 3$ matrix. They proved that such surfaces are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius.

Recently, the surfaces of revolution with no zero Gaussian curvature $K_G$ in the 3-dimensional Lorentz-Minkowski space have been classified under the condition

$$\Delta x^i = \lambda^i x^i,$$

where $\Delta$ is the Laplace operator with respect to the induced metric and $\lambda$ is real number, by Bekkar and Zoubir in [2]. They proved that such surfaces are either minimal or Lorentz hyperbolic cylinders or circular cylinders or pseudospheres of real radius or pseudohyperbolic space of imaginary radius.

In this paper we classify the surfaces of revolution, in the Lorentz-Minkowski space $E^3_1$, under a new condition

$$\tilde{\Gamma}^{1}_{11} \Psi_i = \lambda^i \Psi_i, \quad \lambda^i \in \mathbb{R}, \quad i = 1, 2, 3. \quad (1)$$

Here, $\tilde{\Gamma}^{1}_{11}$ is one of the Christoffel-like operators with non-degenerate metric in $E^3_1$.

2. Preliminaries

In the vector space $\mathbb{R}^n$, if the following Lorentzian inner product

$$\langle, \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x, y) \longrightarrow \langle, \rangle (x, y) = \langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i$$
with 1-index is considered instead of the Euclidean inner product, it is well-
known that the vector space $\mathbb{R}^n$ is a Lorentzian vector space and it is shown by $\mathbb{R}^n_1$. Furthermore, especially if $n = 3$, then this space is called as a Minkowski 3-space and usually shown by $E^3_1$.

Let $M$ be a 2-dimensional surface of the Minkowski 3-space $E^3_1$ equipped with induced metric.

A surface of revolution in Euclidean space is generated by revolving of an arbitrary curve about an arbitrary axis. In Minkowski space, however, there are different types of curves (spacelike, timelike or lightlike (null)) as well as different types of rotation axes (spacelike, timelike or lightlike (null)), so that there are different types of surfaces of revolution in this context [9].

Let $\alpha : I = (a, b) \subset \mathbb{R} \rightarrow \pi$ be a curve in a plane $\pi$ of $E^3_1$ and let $\xi$ be a straight line of $\pi$ which does not intersect the curve $\alpha$. A surface of revolution $M$ in $E^3_1$ is defined as a non-degenerate surface, revolving the curve $\alpha$ around the axis $\xi$.

Now, let us investigate the surfaces of revolution according to the type of the axis of revolution.

First suppose that the axis of revolution is the spacelike axis $x_3 = (0, 0, 1)$. Then, we may assume that the curve $\alpha$ is lying in the $x_2x_3$–plane or $x_1x_3$–plane. So the curve $\alpha$ can be parametrized either by $\alpha(u) = (0, f(u), g(u))$ or $\alpha(u) = (f(u), 0, g(u))$, where $f$, $g$ are smooth functions and $f$ is a positive function.

Let $A$ be a $3 \times 3$ regular matrix and $0 \neq \xi \in E^3_1$ be a vector. If $A$ satisfies the following conditions, then it is said that $A$ denotes a rotation in positive direction [10]:

i) $A\xi = \xi$,
ii) $AIA^t = I$,
iii) $\det A = 1$,

where $I$ is the $3 \times 3$ Lorentzian unit matrix, i. e. ,

$$I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

It can be seen that the rotation matrix which fixes the spacelike axis $x_3 = (0, 0, 1)$ is the set of $3 \times 3$ matrices defined by

$$A(v) = \begin{bmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{bmatrix}, v \in \mathbb{R}.$$
If the curve $\alpha$ is in the $x_2x_3$--plane or $x_1x_3$--plane, then the surface of revolution $M$ can be parametrized by
\begin{equation}
\Psi(u, v) = (f(u) \sinh v, f(u) \cosh v, g(u))
\end{equation}
or
\begin{equation}
\Psi(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u)),
\end{equation}
respectively.

Now, suppose that the axis of revolution is the timelike axis $x_1 = (1, 0, 0)$ and without loss of generality we may assume that the curve $\alpha$ lies in the $x_1x_2$--plane. Then one of its parametrizations is
\[
\alpha(u) = (g(u), f(u), 0),
\]
where $f$, $g$ are smooth functions and $f$ is a positive function.

The rotation matrix which fixes the timelike axis $x_1 = (1, 0, 0)$ is the set of $3 \times 3$ matrices given by
\[
A(v) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos v & -\sin v \\
0 & \sin v & \cos v
\end{bmatrix}, 0 \leq v \leq 2\pi.
\]

So the surface of revolution $M$ around $x_1$ can be parametrized as
\begin{equation}
\Psi(u, v) = (g(u), f(u) \cos v, f(u) \sin v).
\end{equation}

Finally, suppose that the axis of revolution is a lightlike line of plane $x_1x_2$ spanned by the vector $(1, 1, 0)$. Then the rotation matrix which fixes the null axis $(1, 1, 0)$ is the set of $3 \times 3$ matrices given by
\[
A(v) = \begin{bmatrix}
1 + \frac{v^2}{2} & \frac{v^2}{2} & v \\
\frac{v^2}{2} & 1 - \frac{v^2}{2} & v \\
v & -v & 1
\end{bmatrix}, v \in \mathbb{R}.
\]

Thus, if the axis of revolution is the line spanned by the vector $(1, 1, 0)$ and the curve $\alpha(u) = (f(u), g(u), 0)$ lies in the $x_1x_2$--plane, then the surface of revolution $M$ can be parametrized as
\[
\Psi(u, v) = (\left(1 + \frac{v^2}{2}\right)f(u) - \frac{v^2}{2}g(u), \frac{v^2}{2}f(u) + \left(1 - \frac{v^2}{2}\right)g(u), vf(u) - vg(u)).
\]

It is well-known that the $\Gamma^k_{ij}$ Christoffel symbols are defined by
\[
\Gamma^k_{ij} = \frac{1}{2} \sum_m g^{km} \left\{ \frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right\},
\]
where $g^{ij} = (g_{ij})^{-1}$ [11].
Here, we define the following $\tilde{\Gamma}^k_{ij}$ operator which can be written with the aid of $\Gamma^k_{ij}$ Christoffel symbols. And we call it Christoffel-like operators:

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2} \sum_m g^{km} \left\{ g_{jm} \frac{\partial}{\partial x^i} + g_{im} \frac{\partial}{\partial x^j} - g_{ij} \frac{\partial}{\partial x^m} \right\}.$$ 

Let us denote by $E, F, G$ the coefficients of first fundamental forms of these surfaces of revolution. Then, by applying the Christoffel-like operator $\tilde{\Gamma}^1_{11}$ to a smooth function $\Psi(u, v)$, we have

$$\tilde{\Gamma}^1_{11}(\Psi) = \frac{1}{2} \left\{ \frac{G}{EG - F^2} \{ E\Psi_u - \frac{F}{EG - F^2} \{ 2F\Psi_u - E\Psi_v \} \} \right\}.$$ 

(5)

3. The Main Results

In this section, we study the surfaces of revolution $M$ satisfying the condition (1). We distinguish two cases according to whether these surfaces are given by (2) or (4).

**First Case.**

Suppose that the surface of revolution $M$ is given by (2) and the curve $\alpha$ is given by the arc length, so

$$f''(u) + g''(u) = 1, \ \forall u \in I.$$ 

(6)

For this surface of revolution, the coefficients of first and second fundamental forms are

$$\begin{cases} 
E = 1, \ F = 0, \ G = -f^2(u); \\
L = f''(u)g'(u) - f'(u)g''(u), \ M = 0, \ N = f(u)g'(u), 
\end{cases}$$

(7)

respectively, and the mean curvature and Gaussian curvature are

$$\begin{cases} 
H = \frac{1}{2}(f''(u)g'(u) - f'(u)g''(u) - \frac{g'(u)}{f(u)}); \\
K = \frac{-g'(u)}{f(u)}(f''(u)g'(u) - f'(u)g''(u)) = -\frac{f''(u)}{f(u)}, 
\end{cases}$$

(8)

respectively.

Since the relation (6) holds, there exists a smooth function $t = t(u)$ such that

$$f'(u) = \cos t(u), \ g'(u) = \sin t(u), \ \forall u \in I.$$ 

(9)

Hence, we have

$$\begin{cases} 
E = 1, \ F = 0, \ G = -f^2(u), \\
L = -t'(u), \ M = 0, \ N = f(u).\sin t(u), \\
H = \frac{1}{2}(-t'(u) - \frac{\sin t(u)}{f(u)}), \ K = \frac{\sin t(u).t'(u)}{f(u)}. 
\end{cases}$$

(10)
If we apply the condition $\tilde{\Gamma}_{11}^{1} \Psi_{i} = \lambda_{i} \Psi_{i}$ to the surface of revolution (2), we get

$$\begin{aligned}
\tilde{\Gamma}_{11}^{1} (f(u) \sinh v) &= \lambda_{1} f(u) \sinh v \\
\tilde{\Gamma}_{11}^{1} (f(u) \cosh v) &= \lambda_{2} f(u) \cosh v \\
\tilde{\Gamma}_{11}^{1} (g(u)) &= \lambda_{3} g(u).
\end{aligned}$$

(11)

Since $\Psi_{1}(u, v) = f(u) \sinh v$ for the surface of revolution (2), by taking the derivative of $\Psi_{1}(u, v)$ according to $u$ and $v$ parameters we get,

$$\Psi_{1u} = f'(u) \sinh v = \cos t(u) \sinh v, \quad \Psi_{1v} = f(u) \cosh v.$$

On the other hand, since $\tilde{\Gamma}_{11}^{1} \Psi_{i} = \lambda_{i} \Psi_{i}$ for the surface of revolution (2), by using (5), (9), (10) and (11) we get

$$\tilde{\Gamma}_{11}^{1} (f(u) \sinh v) = \frac{1}{2} \cos t(u) \sinh v$$

and so

$$\frac{1}{2} \cos t(u) = \lambda_{1} f(u).$$

Since $\Psi_{2}(u, v) = f(u) \cosh v$ and $\Psi_{3}(u, v) = g(u)$ for the surface of revolution (2), we get from (1)

$$\frac{1}{2} \cos t(u) = \lambda_{2} f(u)$$

and

$$\frac{1}{2} \sin t(u) = \lambda_{3} g(u),$$

respectively.

Therefore, we can write the linear system of equations as

$$\begin{aligned}
\frac{1}{2} \cos t(u) &= \lambda_{1} f(u) \\
\frac{1}{2} \cos t(u) &= \lambda_{2} f(u) \\
\frac{1}{2} \sin t(u) &= \lambda_{3} g(u).
\end{aligned}$$

(12)

From (12), we have $\lambda_{1} = \lambda_{2}$. If we put $\lambda_{1} = \lambda_{2} = \lambda$, and $\lambda_{3} = \mu$, we get the system of equations

$$\begin{aligned}
\cos t(u) &= 2\lambda f(u) \\
\sin t(u) &= 2\mu g(u).
\end{aligned}$$

(13)

So the problem of classifying the surfaces of revolution (2) which satisfy the condition $\tilde{\Gamma}_{11}^{1} \Psi_{i} = \lambda_{i} \Psi_{i}$ is reduced to the integration of system of ordinary differential equations (13).

Now, let us examine the system of equations (13) according to the values of the constants $\lambda$ and $\mu$:

A. Let $\lambda = \mu = 0$. 

Then, we get the system of equations
\[
\begin{align*}
\cos t(u) &= 0 \\
\sin t(u) &= 0.
\end{align*}
\]
But this is not possible. So, in this case there are no surfaces of revolution.

B. Let \( \lambda = \mu \neq 0 \).
In this case we have
\[
\begin{align*}
\cos t(u) &= 2\lambda f(u) \\
\sin t(u) &= 2\lambda g(u).
\end{align*}
\]
(14)
In the system of equations (14), if the first equation is multiplied by 
\(-\sin t(u)\) and the second equation is multiplied by \(\cos t(u)\) and add up the 
resulting equations, we get the equation
\[
f(u) \sin t(u) - g(u) \cos t(u) = 0.
\]
By using (9), we have
\[
f(u)g'(u) - f'(u)g(u) = 0
\]
or equivalently,
\[
\frac{f'(u)}{f(u)} = \frac{g'(u)}{g(u)}.
\]
By integrating the both sides, we get
\[
f(u) = c.g(u).
\]
Hence, the surface of revolution (2) is
\[
\Psi(u, v) = (cg(u) \sinh v, cg(u) \cosh v, g(u)), \ c \in \mathbb{R}.
\]
C. Let \( \lambda = 0, \mu \neq 0 \).
Then, since \( \cos t(u) = 0 \) from (13), we get
\[
\begin{align*}
f'(u) &= \cos t(u) = 0 \text{ or } f(u) = c_1, \ c_1 \in \mathbb{R}; \\
f'^2(u) + g'^2(u) &= 1 \text{ or } g(u) = u + c_2, \ c_2 \in \mathbb{R}.
\end{align*}
\]
Consequently, in this case, the surface of revolution is
\[
\Psi(u, v) = (c_1 \sinh v, c_1 \cosh v, u + c_2).
\]
This surface is Lorentz hyperbolic cylinder \( S_1(\mathcal{r}) \times \mathbb{R} \).

D. Let \( \lambda \neq 0, \mu = 0 \).
Then, from (13) \( \sin t(u) = 0 \) and so from (10), we have \( H = K = 0 \). 
This means that, the surface of revolution is minimal. Furthermore, since
\[
g'(u) = \sin t(u) = 0, \ g(u) = c_1, \ c_1 \in \mathbb{R} \text{ and so } f(u) = u + c_2, \ c_2 \in \mathbb{R}.
\]
Thus the surface of revolution can be parametrized by
\[
\Psi(u, v) = ((u + c_2) \sinh v, (u + c_2) \cosh v, c_1).
\]
E. Let $\lambda \neq 0$, $\mu \neq 0$ and $\lambda \neq \mu$.

In this case the system of equations (13) is
\[
\begin{align*}
\cos t(u) &= 2\lambda f(u) \\
\sin t(u) &= 2\mu g(u).
\end{align*}
\]

Following the same procedure as in the B, one can easily proves that
\[
-\lambda f(u) \sin t(u) + \mu g(u) \cos t(u) = 0,
\]
or equivalently
\[
\frac{f'(u)}{f(u)} = \frac{\lambda g'(u)}{\mu g(u)}.
\]

If we put $\frac{\lambda}{\mu} = c_1$ and integrate the both sides, we have
\[
f(u) = c_2 g^{c_1}(u).
\]

So the surface of revolution can be given by
\[
\Psi(u, v) = (c_2 g^{c_1}(u) \sinh v, c_2 g^{c_1}(u) \cosh v, g(u)), c_1, c_2 \in \mathbb{R}.
\]

Thus, we can give the following theorem:

**Theorem 3.1.** Let $M$ be a surface of revolution given by (2) in $E^3_1$. Then the surface of revolution $M$ satisfies the condition $\tilde{\Gamma}_{11}^i \Psi_i = \lambda_i \Psi_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ if and only if $M$ is one of the following surfaces of revolution:

i) **Surface of revolution**
\[
\Psi(u, v) = (c g(u) \sinh v, c g(u) \cosh v, g(u)), c \in \mathbb{R},
\]

ii) **Lorentz hyperbolic cylinder given by**
\[
\Psi(u, v) = (c_1 \sinh v, c_1 \cosh v, u + c_2), c_1, c_2 \in \mathbb{R},
\]

iii) **Minimal surface given by**
\[
\Psi(u, v) = (u + c_2) \sinh v, (u + c_2) \cosh v, c_1), c_1, c_2 \in \mathbb{R},
\]

iv) **Surface of revolution**
\[
\Psi(u, v) = (c_2 g^{c_1}(u) \sinh v, c_2 g^{c_1}(u) \cosh v, g(u)), c_1, c_2 \in \mathbb{R}.
\]

**Second Case.**

Now, suppose that the surface of revolution $M$ in $E^3_1$ is given by (4). Consider that
\[
f^2(u) - g^2(u) = -1, \forall u \in I.
\]
Because of (15), for a smooth function \( t = t(u) \) we can write
\[
(16) \quad f'(u) = \sinh t(u), \quad g'(u) = \cosh t(u), \quad \forall u \in I.
\]
For this surface, we have
\[
(17) \quad \begin{cases}
E = -1, & F = 0, \quad G = f^2(u), \\
L = g''(u)f'(u) - g'(u)f''(u), & M = 0, \quad N = f(u)g'(u); \\
H = \frac{1}{2}(g'(u)f''(u) - g''(u)f'(u) + \frac{g'(u)}{f(u)}), \\
K = -\frac{g''(u)}{f(u)}(g''(u)f'(u) - g'(u)f''(u)) = \frac{f''(u)}{f(u)}
\end{cases}
\]
and from (16),
\[
(18) \quad \begin{cases}
E = -1, & F = 0, \quad G = f^2(u), \\
L = -t'(u), & M = 0, \quad N = f(u)\cosh t(u), \\
H = \frac{1}{2}(t'(u) + \frac{\cosh t(u)}{f(u)}), & K = \frac{\cosh t(u)\cdot t'(u)}{f(u)}.
\end{cases}
\]
If we apply the condition \( \tilde{\Gamma}_{11}^1 \Psi_i = \lambda_i \Psi_i \) to the surface of revolution (4), we get the system of equations
\[
(19) \quad \begin{cases}
\tilde{\Gamma}_{11}^1(g(u)) = \lambda_1 g(u) \\
\tilde{\Gamma}_{11}^1(f(u) \cos v) = \lambda_2 f(u) \cos v \\
\tilde{\Gamma}_{11}^1(f(u) \sin v) = \lambda_3 f(u) \sin v.
\end{cases}
\]
By using (5), (16), (17) and (19), we get
\[
(20) \quad \begin{cases}
\frac{1}{2}\cosh t(u) = \lambda_1 g(u) \\
\frac{1}{2}\sinh t(u) = \lambda_2 f(u) \\
\frac{1}{2}\sinh t(u) = \lambda_3 f(u).
\end{cases}
\]
So it must be \( \lambda_2 = \lambda_3 \). If we put \( \lambda_2 = \lambda_3 = \lambda \) and \( \lambda_1 = \mu \), we have the system of equations
\[
(20) \quad \begin{cases}
\cosh t(u) = 2\mu g(u) \\
\sinh t(u) = 2\lambda f(u).
\end{cases}
\]
Consequently the problem of classifying the surfaces of revolution (4) which satisfy the condition \( \tilde{\Gamma}_{11}^1 \Psi_i = \lambda_i \Psi_i \) is reduced to the integration of the system of ordinary differential equations (20).

Now let examine the system of equation (20) according to the values of the constants \( \lambda \) and \( \mu \):

A. Let \( \lambda = \mu = 0 \).
Then, from (20), we have the system of equations
\[
\begin{cases}
\sinh t(u) = 0 \\
\cosh t(u) = 0.
\end{cases}
\]
But this is not possible. So, there are no surfaces of revolution when \( \lambda = \mu = 0 \).

B. Let \( \lambda = \mu \neq 0 \).
Then, the system of equations (20) is

\begin{align}
\begin{cases}
\sinh t(u) = 2\lambda f(u) \\
\cosh t(u) = 2\lambda g(u).
\end{cases}
\end{align}

In the system of equations (21), if the first and second equations are multiplied by \(-\cosh t(u)\) and \(\sinh t(u)\), respectively, and add up the resulting equations, we get

\[ f(u) \cosh t(u) - g(u) \sinh t(u) = 0. \]

From (16),

\[ f(u)g'(u) - g(u)f'(u) = 0 \]

or equivalently,

\[ \frac{f'(u)}{f(u)} = \frac{g'(u)}{g(u)}. \]

By integrating both sides, it can be found that

\[ f(u) = c.g(u). \]

Then the surface of revolution (4) is

\[ \Psi(u, v) = (g(u), cg(u) \cos v, cg(u) \sin v), c \in \mathbb{R}. \]

C. Let \(\lambda = 0, \mu \neq 0\).
Since \(\sinh t(u) = 0, f'(u) = \sinh t(u) = 0\) and so \(f(u) = c_1, c_1 \in \mathbb{R}\). Thus we have, \(g(u) = u + c_2, c_2 \in \mathbb{R}\). Hence the surface of revolution when \(\lambda = 0\) and \(\mu \neq 0\) is

\[ \Psi(u, v) = (u + c_2, c_1 \cos v, c_1 \sin v), c_i \in \mathbb{R}, i = 1, 2. \]

This surface is a circular cylinder.

D. Let \(\lambda \neq 0, \mu = 0\).
Then, from (20) \(\cosh t(u) = 0\). This is not possible because of the definition of the function \(\cosh\). So, there are no surfaces of revolution for this case, too.

E. Let \(\lambda \neq 0, \mu \neq 0\) ve \(\lambda \neq \mu\).
Then, the system of equations (20) is

\begin{align}
\begin{cases}
\sinh t(u) = 2\lambda f(u) \\
\cosh t(u) = 2\mu g(u).
\end{cases}
\end{align}

In system of equations (22), if the first equation is multiplied by \(\cosh t(u)\) and the second equation is multiplied by \(-\sinh t(u)\) and add up the resulting equations, we have

\[ \lambda f(u)g'(u) - \mu g(u)f'(u) = 0 \]
or equivalently
\[ \frac{f'(u)}{f(u)} = \frac{\lambda g'(u)}{\mu g(u)}. \]

If we put \( \frac{\lambda}{\mu} = c_1 \), \( c_1 \in \mathbb{R} \) and integrate the both sides, we find that
\[ f(u) = c_2 g^{c_1}(u), \quad c_1, c_2 \in \mathbb{R}. \]

Therefore the surface of revolution is
\[ \Psi(u, v) = (g(u), c_2 g^{c_1}(u) \cos v, c_2 g^{c_1}(u) \sin v), \quad c_1, c_2 \in \mathbb{R}. \]

Thus, we can give the following theorem:

**Theorem 3.2.** Let \( M \) be a surface of revolution given by (4) in \( E_1^3 \). Then the surface of revolution \( M \) satisfies the condition \( \tilde{\Gamma}_{11}^1 \Psi_i = \lambda_i \Psi_i \), \( \lambda_i \in \mathbb{R} \), \( i = 1, 2, 3 \) if and only if \( M \) is one of the following surfaces of revolution:

i) **Surface of revolution**
\[ \Psi(u, v) = (g(u), c g(u) \cos v, c g(u) \sin v), \quad c \in \mathbb{R}, \]

ii) **Circular cylinder given by**
\[ \Psi(u, v) = (u + c_2, c_1 \cos v, c_1 \sin v), \quad c_1, c_2 \in \mathbb{R}, \]

iii) **Surface of revolution**
\[ \Psi(u, v) = (g(u), c_2 g^{c_1}(u) \cos v, c_2 g^{c_1}(u) \sin v), \quad c_1, c_2 \in \mathbb{R}. \]

Now, let us give some examples for these surfaces of revolution:

**Example 3.1.** Let us determine the type of the surface
\[ \Psi(u, v) = (3 \sinh v, 3 \cosh v, u + 2) \]
according to the values of \( \lambda \) and \( \mu \) under the condition \( \tilde{\Gamma}_{11}^1 \Psi_i = \lambda_i \Psi_i \):

Derivatives according to \( u \) and \( v \) of the surface \( \Psi(u, v) \), respectively, are
\[ \Psi_u = (0, 0, 1), \quad \Psi_v = (3 \cosh v, 3 \sinh v, 0). \]

Furthermore,
\[ E = 1, F = 0, G = -9. \]

Since
\[ \begin{cases} \tilde{\Gamma}_{11}^1(3 \sinh v) = 0, \\ \tilde{\Gamma}_{11}^1(3 \cosh v) = 0, \\ \tilde{\Gamma}_{11}^1(u + 2) = \frac{1}{2}, \end{cases} \]
we have from (1),
\[ \begin{cases} 
0 = \lambda_1 \sinh v, \\
0 = \lambda_2 \cosh v, \\
\frac{1}{2} = \lambda_3(u + 2).
\end{cases} \]

Therefore it must be \( \lambda_1 = \lambda_2 = \lambda = 0 \text{ and } \lambda_3 = \mu \neq 0 \). Then, this situation shows that this surface is belong to the First Case’s C classy and so it is Lorentz hyperbolic cylinder. Also, for this surface,
\[ L = 0, M = 0, N = 3 \text{ and } H = -\frac{1}{6}, K = 0. \]

**Example 3.2.** For the surface of 
\[ \Psi(u, v) = ((u + 2) \sinh v, (u + 2) \cosh v, 3), \]
we find
\[ E = 1, F = 0, G = -(u + 2)^2. \]

Since \( \tilde{\Gamma}_{11}^1((u + 2) \sinh v) = \frac{1}{2} \sinh v, \tilde{\Gamma}_{11}^1((u + 2) \cosh v) = \frac{1}{2} \cosh v \) and \( \tilde{\Gamma}_{11}^1(3) = 0, \)
we have from (1), \( \frac{1}{2} = \lambda_1(u + 2), \frac{1}{2} = \lambda_2(u + 2) \text{ and } 0 = 3\lambda_3, \) respectively. So it must be \( \lambda_1 = \lambda_2 = \lambda \neq 0 \text{ and } \lambda_3 = \mu = 0 \) (First Case’s D classy).
Here,
\[ L = M = N = 0 \text{ and } H = K = 0. \]

**Example 3.3.** Let us investigate the surface 
\[ \Psi(u, v) = (\cos u, 3 \cos^2 u \cos v, 3 \cos^2 u \sin v). \]
Since
\[ \begin{cases} 
\tilde{\Gamma}_{11}^1(\cos u) = -\frac{1}{2} \sin u \\
\tilde{\Gamma}_{11}^1(3 \cos^2 u \cos v) = -3 \cos u. \sin u. \cos v \\
\tilde{\Gamma}_{11}^1(3 \cos^2 u \sin v) = -3 \cos u. \sin u. \sin v
\end{cases} \]
we get from (1),
\[
\begin{align*}
-\frac{1}{2} \sin u &= \lambda_1 \cos u \\
- \sin u &= \lambda_2 \cos u \\
- \sin u &= \lambda_3 \cos u.
\end{align*}
\]
Hence, \( \lambda_2 = \lambda_3 = \lambda \neq 0 \) and \( \lambda_1 = \mu \neq 0 \). And also \( \lambda \neq \mu \). So, this surface is belong to Second Case’s E classy. This surface can be seen in Figure 2.

\[
\text{Figure 2. } \Psi(u,v)=(\cos u,3\cos^2 u\cos v,3\cos^2 u\sin v)
\]

**Example 3.4.** If we consider the surface of
\[
\Psi(u, v) = (u + 2, 3 \cos v, 3 \sin v)
\]
and apply (1), we have
\[
\begin{align*}
\frac{1}{2} &= \lambda_1 (u + 2) \\
0 &= 3\lambda_2 \cos v \\
0 &= 3\lambda_3 \sin v.
\end{align*}
\]
So, \( \lambda_2 = \lambda_3 = \lambda = 0 \) and \( \lambda_1 = \mu \neq 0 \). Then, this surface of revolution is belong to Second Case’s C classy.

**References**


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