On the $\overline{M}$-Integral Curves and $\overline{M}$-Geodesic Sprays in Minkowski Space

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Abstract
In this study $\overline{M}$-vector field $Z$ on $M$, $\overline{M}$-integral curve of $Z$, and $\overline{M}$-geodesic spray concepts are given, where $\overline{M}$ is a Lorentzian manifold in Minkowski space and $M$ is a hypersurface of $\overline{M}$. Then, the following main theorem related with these concepts is given and proved: The natural lift $\pi$ of the curve $\alpha$ is an $\overline{M}$-integral curve of $\overline{M}$-geodesic spray $Z$ if and only if $\alpha$ is an $\overline{M}$-geodesic on $M$.

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Let $\overline{M}$ be a Lorentzian manifold and $M$ be a Lorentzian hypersurface of $\overline{M}$. $\overline{D}$ being the Levi-Civita connection on $\overline{M}$, $S$ being Weingarten map of $M$ and $N$ being unit normal vector field on $M$ we have the Gauss Equation given by

$$\overline{D}_XY = D_XY + \varepsilon g(S(X), Y)N \tag{1}$$

where $D$ is the Levi-Civita connection on $M$. 
Definition 1  Let $Z$ be a vector field on $\mathcal{M}$. $Z$ is called an $\mathcal{M}$-vector field on $M$ if $Z$ is a mapping which attaches to each point $P$ in $M$, a vector $Z_P$ in $T_PM$, that is

$$Z : M \longrightarrow T_PM.$$ 

Any $\mathcal{M}$-vector field $Z$ can be decomposed into its tangential and normal components given by

$$Z = Z_t + Z_n$$

where $Z_t$ is a tangent vector field on $M$ and $Z_n$ is a vector field on $\mathcal{M}$ defined on $M$ which is normal to $M$ at every point. We have

$$Z = Z_t + \lambda N$$

where $\lambda \in C^\infty (M, \mathbb{R})$.

Let $\alpha$ be a curve passing through a point $P$ on $M$ and $T$ denote the tangent vector field of $\alpha$ on $M$. Covariant derivative of $Z$ in the direction $T$ gives

$$\overline{D}_TZ = \overline{D}_TZ_t + \overline{D}_T\lambda N$$

and then

$$\overline{D}_TZ = D_TZ_t + \varepsilon g (S (T), Z_t) N + D_T\lambda N + \varepsilon g (S (T), \lambda N) N.$$ 

After some calculations we have

$$\overline{D}_TZ = \tan \overline{D}_TZ + \text{nor} \overline{D}_TZ$$

(3)

where

$$\overline{D}_TZ = D_TZ_t - \lambda S (T), \text{nor} \overline{D}_TZ = \left( \frac{d\lambda}{dt} + \varepsilon g (S (T), Z_t) \right) N.$$ 

(4)

Definition 2  The vector $\overline{D}_TZ$, $\tan \overline{D}_TZ$, and $\text{nor} \overline{D}_TZ$ in (3) are called the absolute curvature vector, geodesic curvature vector, and normal curvature vector of $\mathcal{M}$-vector field $Z$ with respect to $\alpha$, respectively and the corresponding magnitudes on $M$ are called the absolute curvature, geodesic curvature and normal curvature of the $\mathcal{M}$-vector field $Z$ with respect to $\alpha$. Hence
\[
\overline{K}_{ZA} = \|D_T Z\| \iff D_T Z = \overline{K}_{ZA} \overline{N}_A
\]

\[K_{ZG} = \|\tan D_T Z\| \iff \tan D_T Z = K_{ZG} X \tag{5}\]

\[K_{ZN} = \|\text{nor } D_T Z\| \iff \text{nor } D_T Z = K_{ZN} N \]

where \(\overline{N}_A\) is a unit vector field on \(\overline{M}\), \(X\) is a unit vector field on \(M\), \(\overline{K}_{ZA}\), \(K_{ZG}\) and \(K_{ZN}\) are absolute curvature, geodesic curvature, and normal curvature respectively.

**Definition 3** Let \(M\) be a Lorentzian hypersurface of \(\overline{M}\), \(D\) be the connection on \(M\) and \(\overline{D}\) be the connection on \(\overline{M}\). If \(\tan D_T Z = 0\) the curve \(\alpha\) is called \(\overline{M}\)-geodesic on \(M\), where \(T\) is the unit tangent vector of the curve \(\alpha : I \to M\).

**Definition 4** A vector \(X \in T_PM\) is called as an asymptotic vector of \(Z\) if

\[
\left( \frac{d\lambda}{dt} + \varepsilon g (S (X), Z_t) \right) \left( \frac{d\lambda}{dt} + \varepsilon g (S (X), Z_t) \right) = 0. \tag{6}\]

The curve \(\alpha\) on \(M\) is called the asymptotic curve of \(Z\) if the tangent vector field of \(\alpha\) coincides with the asymptotic vector field of \(Z\), that is,

\[
\left( \frac{d\lambda}{dt} + \varepsilon g (S (T), Z_t) \right) \left( \frac{d\lambda}{dt} + \varepsilon g (S (T), Z_t) \right) = 0,
\]

where \(T = \frac{d\alpha}{dt}\).

**Definition 5** \(X, Y \in T_PM\) are called conjugate vectors of \(Z\) if

\[
\left( \frac{d\lambda}{dt} + \varepsilon g (S (X), Z_t) \right) \left( \frac{d\lambda}{dt} + \varepsilon g (S (Y), Z_t) \right) = 0. \tag{7}\]

\(X\) is called a self-conjugate vector field of \(Z\) if

\[
\left( \frac{d\lambda}{dt} + \varepsilon g (S (X), Z_t) \right) \left( \frac{d\lambda}{dt} + \varepsilon g (S (X), Z_t) \right) = 0. \tag{8}\]
Using **Definition 5** we have obtained the results:

**Corollary 1.1** Tangent vector field of every asymptotic curve of $Z$ is a self-conjugate vector field of $Z$.

**Corollary 1.2** If $X$ is an asymptotic vector field of $Z$ then for the value of $\lambda$ in (2) we have

$$\lambda = - \int \varepsilon g(S(X), Z_t) \, dt. \quad (9)$$

**Definition 6** For an $\overline{M}$-vector field $Z = Z_t + Z_n$, a curve $\alpha \subset M$ is called an $\overline{M}$-integral curve of $Z$ if

$$Z_t (\alpha(t)) = \frac{d\alpha}{dt} |_{\alpha(t)}. \quad (10)$$

**Definition 7** Let $\alpha \subset M$ be a differentiable curve. The curve $\overline{\alpha} : I \rightarrow TM$ given by

$$\overline{\alpha}(t) = \dot{\alpha}(t) |_{\alpha(t)} \quad (11)$$

is called the natural lift of $\alpha$ on the manifold $TM$.

**Definition 8** A $\overline{M}$-vector field $Z$ is called an $\overline{M}$-geodesic spray if for $V \in TM$

$$Z_t (V) = \left( \frac{d\lambda}{dt} + \varepsilon g(S(V), V) \right) N. \quad (12)$$

**Theorem 1.3** The natural lift $\overline{\alpha}$ of the curve $\alpha$ is an $\overline{M}$-integral curve of $\overline{M}$-geodesic spray $Z$ if and only if $\alpha$ is an $\overline{M}$-geodesic on $M$.

**Proof.** Let $\overline{\alpha}$ be an $\overline{M}$-integral curve of the $\overline{M}$-geodesic spray $Z$. Then

$$Z_t (\overline{\alpha}) = \frac{d\overline{\alpha}}{dt}. \quad (13)$$

Since $Z$ is a geodesic spray on $T\overline{M}$ ($\overline{M}$-geodesic spray) we have

$$Z_t (\overline{\alpha}) = \left( \frac{d\lambda}{dt} + \varepsilon g(S(\overline{\alpha}), \overline{\alpha}) \right) N. \quad (14)$$

Joining (11), (13) and (14) we obtain

$$\frac{d\overline{\alpha}}{dt} = \left( \frac{d\lambda}{dt} + \varepsilon g \left( S \left( \dot{\alpha}, \dot{\alpha} \right) \right) \right) N.$$
On the other hand, since
\[
\frac{d\alpha}{dt} = \frac{d\dot{\alpha}}{dt} = \mathcal{D}_\alpha \dot{\alpha},
\]
using (3) we have
\[
\mathcal{D}_\alpha \dot{\alpha} - \lambda S \left( \dot{\alpha} \right) = 0.
\]
This shows that \( \alpha \) is an \( \vec{M} \)-geodesic on \( M \) which is to be shown.

Conversely, if \( \alpha \) is an \( \vec{M} \)-geodesic on \( M \) then it is obvious that the natural lift \( \vec{\sigma} \) is an \( \vec{M} \)-integral curve of the \( \vec{M} \)-geodesic spray \( Z \).

References


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