Deformation of a Trans-Sasakian Manifold

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Abstract

In this paper we shall show trans-Sasakian manifold is invariant under some deformation with an example in 3-dimensional trans-Sasakian manifold. Also we shall discuss some properties on trans-Sasakian manifold with the deformation and the Nijenhuis tensor on trans-Sasakian manifold is invariant with respect to the same deformation.

Mathematics Subject Classification: 53C05, 53C15

Keywords: Trans-Sasakian manifold, Deformation, Nijenhuis tensor
1. Introduction

In 1985, J.A. Oubina [5] introduced a new class of almost contact manifold namely trans-Sasakian manifold and J.C Marrero studied local structure of trans-Sasakian manifolds [6]. K. Yano studied deformation of a Sasakian manifold [7]. Here we shall apply some particular type of $D$-deformation on trans-Sasakian manifold.

Let $M$ be an almost contact metric manifold of dimension $(2n+1)$ with an almost contact metric structure $(\phi, \xi, \eta, g)$ where $\phi$ is a $(1,1)$ tensor field, $\xi$ is contravariant vector field, $\eta$ is a $1$-form and $g$ is an associated Riemannian metric such that,

\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \\
\eta(\xi) &= 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad (3) \\
g(X, \phi Y) &= -g(\phi X, Y), \quad (4)
\end{align*}

and

\begin{equation}
g(X, \xi) = \eta(X), \tag{5}
\end{equation}

$\forall \ X, Y \in T(M)$, then $M$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$ provided,

\begin{equation}
(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{6}
\end{equation}

holds, for smooth functions $\alpha$ and $\beta$ on $M$ [8], [1], [4].

From (6), it can be shown that,

\begin{equation}
\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi). \tag{7}
\end{equation}

Let $(M, \phi, \xi, \eta, g)$ be a trans-Sasakian manifold and $\mu$ be an automorphism, where $\mu = a\xi$ for some real $a$ such that $1 + a > 0$. $D$-deformation is defined in [7], as $D$ be the distribution defined by $\eta = 0$ along with $\mu = a\xi$. 
2. Some results on trans-Sasakian manifold under $D$-deformation:

In this section we prove:

**Theorem 2.1:** In a trans-Sasakian manifold $(M, \phi, \xi, \eta, g)$ the following relations hold:

\[ L_\mu g = 2\beta g(\phi X, \phi Y), \]  
\[ [\mu, \xi] = 0, \]  
\[ 1 + \eta(\mu) > 0, \]

where $L_\mu$ is the Lie differentiation with respect to $\mu$.

**Proof:** $(M, \phi, \xi, \eta, g)$ is a trans-Sasakian manifold and $\mu$ be a vector field over $M$, where $\mu = a\xi$.

Then $(L_\mu g)(X, Y) = \mu g(X, Y) - g(\nabla_\mu X, Y) - g(X, \nabla_\mu Y)$,

\[ = a\xi g(X, Y) - g([a\xi, X], Y) - g(X, [a\xi, Y]). \]

Again $T(\xi, X) = \nabla_\xi X - \nabla_X \xi - [\xi, X] = 0$ implies

\[ [\xi, X] = \nabla_\xi X - \nabla_X \xi. \]  

So, $L_\mu g(X, Y) = a\{\xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y])\}$,

\[ = a(\nabla_\xi g)(X, Y) + a[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)]. \]

Using (7), we get

\[ L_\mu g(X, Y) = a[g(-\alpha \phi X + \beta(X - \eta(X)\xi), Y) \]  
\[ + g(X, -\alpha \phi Y + \beta(Y - \eta(Y)\xi))], \]

since$(\nabla_\xi g)(X, Y) = 0$.

Again using (4), (5) and (3), we obtain

\[ L_\mu g = 2\beta g(\phi X, \phi Y). \]

Next, $[\mu, \xi] = [a\xi, \xi] = a[\xi, \xi] = 0$

and $1 + \eta(\mu) = 1 + \eta(a\xi) = 1 + a\eta(\xi) = 1 + a > 0$.

**Corollary 2.1:** $L_\mu g = 2\beta g(\phi X, \phi Y)$ is the general form of the first condition, $L_\mu g = 0$, of Theorem 7.2 in [7].
Let us consider the structure after $D$-deformation denoted by $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are defined by

\[
\tilde{\phi}(X) = \phi(X - \tilde{\eta}(X)\tilde{\xi}), \tag{12}
\]

\[
\tilde{\eta} = (1 + \eta(\mu))^{-1}\eta, \tag{13}
\]

\[
\tilde{\xi} = \xi + \mu \tag{14}
\]

and

\[
\tilde{g}(X,Y) = (1 + \eta(\mu))^{-1}g(X - \tilde{\eta}(X)\tilde{\xi}, Y - \tilde{\eta}(Y)\tilde{\xi}) + \tilde{\eta}(X)\tilde{\eta}(Y), \tag{15}
\]

where $X$ and $Y$ are vector fields over $M$ [7].

**Theorem 2.2:** Let $D$ be the deformation on a trans-Sasakian manifold $(M, \phi, \xi, \eta, g)$, then $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also a trans-Sasakian manifold.

**Proof:** Since $D$ is the distribution defined by $\eta = 0$, i.e., for any $X \in D$, $\eta(X) = 0$.

Now we shall show the properties of trans-Sasakian manifold with respect to the deformation $\mu$.

Using (13) and (14), we get

\[
\tilde{\eta}(\tilde{\xi}) = (1 + \eta(\mu))^{-1}\eta(\xi + \mu)
\]

\[
= (1 + \eta(\mu))^{-1}\eta(\xi) + (1 + \eta(\mu))^{-1}\eta(\mu)
\]

\[
= (1 + \eta(\mu))^{-1}(1 + \eta(\mu)), \text{ since } \eta(\xi) = 1.
\]

or,

\[
\tilde{\eta}(\tilde{\xi}) = 1. \tag{16}
\]

Next using (12), (14) and (16), we obtain

\[
\tilde{\phi}(\tilde{\xi}) = \phi(\tilde{\xi} - \tilde{\eta}(\tilde{\xi})\tilde{\xi}) = \phi(\tilde{\xi} - \tilde{\xi}) \text{ as } \tilde{\eta}(\tilde{\xi}) = 1,
\]

or,

\[
\tilde{\phi}(\tilde{\xi}) = 0. \tag{17}
\]

Let $X$ be a vector field, which belongs to $D$, then

\[
\tilde{\eta}(X) = 0. \tag{18}
\]
For $X, Y \in D$ we have from above definitions (12), (13), (14) and (15)

$$\tilde{\phi}(X) = \phi(X),$$  \hspace{1cm} (19)

$$\tilde{\eta} = (1 + \eta(\mu))^{-1}\eta,$$  \hspace{1cm} (20)

$$\tilde{\xi} = \xi + \mu$$  \hspace{1cm} (21)

and

$$\tilde{g}(X, Y) = (1 + \eta(\mu))^{-1}g(X, Y).$$  \hspace{1cm} (22)

Now using (13) and (19),

$$\tilde{\eta}(\tilde{\phi})(X) = \tilde{\eta}\phi(X)$$  

$$= (1 + \eta(\mu))^{-1}\eta\phi(X) = 0,$$  

$$\tilde{\eta}(\tilde{\phi})(X) = 0,$$  \hspace{1cm} (23)

using (19) we have

$$\tilde{\phi}^2(X) = \tilde{\phi}\phi(X),$$  

$$= \phi(\phi(X))$$  

$$= \phi^2(X) = -X.$$  

Thus using (18) we get

$$\tilde{\phi}^2(X) = -X + \tilde{\eta}(X)\tilde{\xi},$$  \hspace{1cm} (24)

According to the definition of $\tilde{g}$ and from (23) we have

$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = (1 + \eta(\mu))^{-1}g(\tilde{\phi}X - \eta(\tilde{\phi}X)\tilde{\xi}, \tilde{\phi}Y - \eta(\tilde{\phi}Y)\tilde{\xi})$$  

$$+ \eta(\tilde{\phi}X)\eta(\tilde{\phi}Y)$$  

$$= (1 + \eta(\mu))^{-1}g(\phi X, \phi Y),$$  \hspace{1cm} using (23).

Since $\eta = 0$, using (3) and (19), we get

$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y).$$  \hspace{1cm} (25)

Again,

$$\tilde{g}(\tilde{\phi}X, Y) + \tilde{g}(X, \tilde{\phi}Y) = g(\phi X, Y) + g(X, \phi Y) = 0.$$  \hspace{1cm} (26)

Replacing $Y$ by $\tilde{\xi}$ in (15), we obtain

$$\tilde{g}(X, \tilde{\xi}) = (1 + \eta(\mu))^{-1}g(X - \tilde{\eta}(X)\tilde{\xi}, \tilde{\xi} - \tilde{\eta}(\tilde{\xi})\tilde{\xi}) + \tilde{\eta}(X)\tilde{\eta}(\tilde{\xi})$$
\[(1 + \eta(\mu))^{-1}g(X, \tilde{\xi} - \tilde{\xi}) + \tilde{\eta}(X), \text{ [by using (16)]}\]

\[\tilde{g}(X, \tilde{\xi}) = \tilde{\eta}(X). \quad (27)\]

For \(X, Y \in D\) in (6) we get,

\[(\nabla_X \phi)Y = [\alpha g(X, Y) + \beta g(\phi X, Y)]\xi. \quad (28)\]

\[(\nabla_X \tilde{\phi})Y = \nabla_X \tilde{\phi}(Y) - \tilde{\phi}(\nabla_X Y),\]

\[= \nabla_X \phi(Y) - \phi(\nabla_X Y),\]

\[(\nabla_X \tilde{\phi})Y = (\nabla_X \phi)Y. \quad (29)\]

Now,

\[\alpha(\tilde{g}(X, Y)\tilde{\xi} - \tilde{\eta}(Y)X) + \beta(\tilde{g}(\tilde{\phi} X, Y)\tilde{\xi} - \tilde{\eta}(Y)\tilde{\phi} X)\]

\[= \alpha \tilde{g}(X, Y)\tilde{\xi} + \beta \tilde{g}(\tilde{\phi} X, Y)\tilde{\xi}\]

\[= (1 + \eta(\mu))^{-1}[\alpha g(X, Y) + \beta g(\phi X, Y)](\xi + \mu)\]

\[= (1 + \eta(\mu))^{-1}(1 + a)[\alpha g(X, Y) + \beta g(\phi X, Y)]\xi\]

\[= (1 + \eta(\mu))^{-1}(1 + a)(\nabla_X \tilde{\phi})Y, \text{ using (28) and (29)}\]

or,

\[(\nabla_X \tilde{\phi})Y = \alpha(\tilde{g}(X, Y)\tilde{\xi} - \tilde{\eta}(Y)X) + \beta(\tilde{g}(\tilde{\phi} X, Y)\tilde{\xi} - \tilde{\eta}(Y)\tilde{\phi} X). \quad (30)\]

Hence (16), (17), (23), (24), (25), (26), (27) and (30) shows that \((M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) is a trans-Sasakian manifold.

3. The Nijenhuis tensor in the trans-Sasakian manifold with respect to the \(D\)-deformation.: The Nijenhuis tensor in the trans-Sasakian manifold is defined by

\[N(X, Y) = [X, Y] + \phi[X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] - \{X \eta(Y) - Y \eta(X)\}\xi. \quad (31)\]

where \(X, Y \in M\) [7].

**Theorem 3.1:** The Nijenhuis tensor in trans-Sasakian manifold is invariant with respect to \(D\) deformation.
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**Proof:** Let $X$ and $Y$ be the vector fields in $D$.

Then $\eta(X) = \eta(Y) = 0$. Also $\tilde{\eta}(X) = \tilde{\eta}(Y) = 0$.

For $X, Y \in D$ in (31), we get,

$$N(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y].$$

Under the $D$-deformation the Nijenhuis tensor becomes

$$\tilde{N}(X, Y) = [X, Y] + \tilde{\phi}[\tilde{\phi} X, Y] + \tilde{\phi}[X, \tilde{\phi} Y] - [\tilde{\phi} X, \tilde{\phi} Y],$$

using (19)

$$= [X, Y] + \tilde{\phi}[\phi X, Y] + \tilde{\phi}[X, \phi Y] - [\phi X, \phi Y]$$

or, $\tilde{N}(X, Y) = N(X, Y)$.

4. Example of a 3-dimensional trans-Sasakian manifold which remains invariant under $D$-deformation:

Example of a 3-dimensional trans-Sasakian manifold is given in [2], where $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, $(x, y, z)$ being coordinates in $\mathbb{R}^3$ and $E_1 = e^z \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right)$, $E_2 = e^z \frac{\partial}{\partial y}$ and $E_3 = \frac{\partial}{\partial z}$.

The Riemannian metric $g$ defined by

$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0,$$

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.$$

Let $\eta$ be 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$.

Using the linearity of $\phi$ and $g$ we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U) E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U) \eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Authors have shown that $(\phi, \xi, \eta, g)$ is a trans-Sasakian structure on $M$. Now under the deformation $D$,

$$\tilde{\phi} E_1 = \phi(E_1 - \tilde{\eta}(E_1) \xi)$$

using (14), (15) and since $1 + a > 0$ we obtain

$$\tilde{\phi} E_1 = \phi(E_1 - \eta(E_1) \xi)$$

and as $\eta(U) = g(U, E_3)$ we get

$$\tilde{\phi} E_1 = \phi(E_1) = E_2, \ [\text{since } g(E_1, E_3) = 0].$$
Similarly we have
\[ \tilde{\phi}E_2 = -E_1 \text{ and } \tilde{\phi}E_3 = 0. \]

So under $D$-deformation trans-Sasakian structure on $M$ remains invariant.

**References**


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Received: April, 2011