M-Projective Curvature Tensor on Kaehler Manifold

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Abstract

Properties of $M$ – projective curvature tensor been studied on Kaehler manifold with respect to recurrent and symmetric properties.

1. Introduction

We consider a $2n$ – dimensional Kaehler manifold $M_{2n}$ with a vector valued linear function $F$ and a Riemannian metric $g$ which satisfies the following conditions:

1.1) $\bar{X} = -I_{2n}$, where $F(X) = \bar{X}$.
1.2) $g(\bar{X}, \bar{Y}) = g(X, Y)$.
1.3) $F(X, T) = g(\bar{X}, Y)$.
1.4) $(D_{a} F)Y = 0$, where $D$ is the Riemannian connection.

If we define [1]

1.5) a) $H(Y, Z) = -\frac{1}{2} \bar{C} \bar{R}(Y, Z)$, b) $H(\bar{Y}, Z) = -H(Y, \bar{Z})$

Then we have

1.6) a) $H(Y, Z) = S(Y, Z)$, b) $H(\bar{Y}, Z) = -H(Y, \bar{Z}) = S(Y, Z)$

where $R$ and $S$ are the so called Riemannian curvature tensor and the Ricci tensor respectively.
The projective curvature tensor $W$, Conformal curvature tensor $C$, Conharmonic curvature tensor $L$, Conircular curvature tensor $V$, $H$ – projective curvature tensor $P$, $H$ – conharmonic curvature tensor $T$, $H$ – Concircular curvature tensor $K$, Conharmonic* curvature tensor $T^*$, $H$ – conformal (Bochner) curvature tensor $B$ and the conharmonic curvature tensor $C^*$ are given on Kaehler manifold respectively by;

1.7) a) $W(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(2n-1)}[S(\gamma',Z)X - S(X,Z)\gamma']$

b) $C(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(2n-1)}[S(\gamma',Z)X - S(X,Z)\gamma'] - g(\gamma',Z)RX + \frac{r}{2(2n-1)}[g(\gamma',Z)X - g(X,Z)\gamma']$

c) $L(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(2n-1)}[S(\gamma',Z)X - S(X,Z)\gamma'] + g(\gamma',Z)RX$

d) $V(X,Y,Z) = R(X,Y,Z) - \frac{r}{2(2n-1)}[g(\gamma',Z)X - g(X,Z)\gamma']$

e) $P(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(2n+2)}[S(\gamma',Z)X - S(X,Z)\gamma'] + S(\gamma',Z)\bar{X}$

f) $T(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(2n+2)}[S(\gamma',Z)X - S(X,Z)\gamma'] - g(\gamma',Z)RX + \frac{1}{2}F(\gamma',Z)RX$

g) $K(X,Y,Z) = R(X,Y,Z) - \frac{r}{4n+1}[g(\gamma',Z)X - g(X,Z)\gamma'] + \frac{1}{2}F(\gamma',Z)\bar{X}$

h) $T^*(X,Y,Z) = R(X,Y,Z) + \frac{r}{2(2n-1)(2n-1)}[g(\gamma',Z)X - g(X,Z)\gamma']$

i) $B(X,Y,Z) = R(X,Y,Z) - \frac{1}{2(2n+2)}[S(\gamma',Z)X - S(X,Z)\gamma'] - g(\gamma',Z)RX + S(\gamma',\bar{Z})\gamma' - S(\gamma',Z)\bar{X} - \frac{1}{2}F(\gamma',Z)RX$

$+ \frac{1}{2}F(\gamma',Z)\bar{X} - 2\frac{1}{2}F(\gamma',Z)R\bar{X} + 2S(X,\gamma',\bar{Z})$
**M-projective curvature tensor on Kaehler manifold**

\[
\frac{r}{4(n+1)(n+2)}\left[ g(X,Z)\overline{g} - g(Y,Z)X - \frac{1}{2(n-1)}[g(Y,Z)RX - g(X,Z)RY] \right] + \frac{r}{2(n-1)(2n-1)}[g(Y,Z)X - g(X,Z)\overline{Y}] \\
\]

where \( r \) is the scalar curvature tensor.

A Kaehler manifold is said to be recurrent if for a non-zero recurrence vector \( v \), that satisfies:

\[ (D_v)R (X,Y,Z) = v (U)R (X,Y,Z) \]

From which we have:

\[ (D_v)S (Y,Z) = v (U)S (Y,Z) \]

Also on a Kaehler manifold \( \overline{1}H \) is said to be recurrent if it satisfies:

\[ (D_v)\overline{1}H (X,Y) = v (U) \overline{1}H (X,Y) \]

A Kaehler manifold is said to be symmetric if it satisfies:

\[ (D_v)R (X,Y,Z) = 0 \]

From which we have:

\[ (D_v)S (Y,Z) = 0 \]

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\[ (D_v)\overline{1}H (X,Y) = v (U) \overline{1}H (X,Y) \]

A Kaehler manifold is said to be symmetric if it satisfies:

\[ (D_v)R (X,Y,Z) = 0 \]
2. M-projective recurrent Kaehler manifold

A Kaehler manifold is said to be $M$ – projective recurrent if it satisfies. 

2.1) \((D_u M)(X,Y,Z) = v(U)M(X,Y,Z)\)

For a non zero recurrence vector $v$. From 1.17 we have;

2.2) \((D_u M)(X,Y,Z) = v(U)M(X,Y,Z) = (D_u R)(X,Y,Z)\)

\[-v(U)R(X,Y,Z) - \frac{1}{4(n-1)}[((D_v S)(Y,Z) - v(U)S(Y,Z))X\]


\[-v(U)S(Y,Z)X + ((D_v S)(X,Z) - v(U)S(X,Z)Y]\]

If the manifold is $M$ – projective recurrent we have;

2.3) \((D_u R)(X,Y,Z) = v(U)R(X,Y,Z) = \frac{1}{4(n-1)}[((D_v S)(Y,Z)\]

\[-v(U)S(Y,Z)X - ((D_v S)(X,Z) - v(U)S(X,Z)Y\]

\[-((D_v S)(Y,Z) - v(U)S(Y,Z))X + ((D_v S)(X,Z)\]

\[-v(U)S(X,Z)Y] = 0\]

Contracting this equation with respect to $X$ we get;

2.4) \[\frac{n-2}{2(n-1)}((D_u S)(Y,Z) - v(U)S(Y,Z) = 0\]

Hence we can state:

**Theorem 2.1:** A $M$ – projective recurrent Kaehler manifold $M_{2n}, n > 2$ is Ricci recurrent.

**Theorem 2.2:** A Kaehler manifold $M_{2n}, n > 2$ $M$ – projective current if and only if it is recurrent.

**Theorem 2.3:** A Flat Kaehler manifold is $M$ – projective recurrent if any only if it is Ricci recurrent.

**Theorem 2.4:** If on a Kaehler manifold two of the following hold, the third also hold.

a. It is $M$ – projective recurrent manifold.

b. It is recurrent manifold.

c. It is Ricci recurrent manifold.

Now barring $Z$ in 2.4 and using 1.6.a we get:

2.5) \[\frac{n-2}{2(n-1)}((D_u)^1H(Y,Z) - v(U)^1H(Y,Z) = 0\]
Hence, we have;

**Theorem 2.5:** On an $M$–projective recurrent Kaehler manifold $M_{2n}$, $n > 2$, $H$ is recurrent.

From 1.17 and 1.7.d we have;

$$2.6) M(X,Y,Z) = V(X,Y,Z) - \frac{1}{4(n-1)}[S(Y,Z)X - S(Y',Z)Y' - S(Y',Z')X']$$

$$+ S(X,Z)\bar{Y} + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y']$$

From which we have:

$$2.7) (D_vM)(X,Y,Z) - v(U)M(X,Y,Z) = (D_vV)(X,Y,Z)$$

$$- v(U)W(X,Y,Z) - \frac{1}{4(n-1)}[(D_vS)(Y,Z) - v(U)S(Y,Z)]X$$

$$- ((D_vS)(X,Z) - v(U)S(X,Z))Y' - ((D_vS)(Y,Z) - v(U)S(Y,Z))\bar{X}$$

$$+ ((D_vS)(X,Z) - v(U)S(X,Z))\bar{Y}' + \frac{(D_vr - v(U)r)}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y']$$

Hence, we can state:

**Theorem 2.6:** On a Kaehler manifold if any two of the following hold, the third also hold:

a. It is $M$–projective recurrent manifold.

b. It is Concircular recurrent manifold.

c. It is a Ricci recurrent manifold.

Similarly we can prove nine theorems analog to theorem 2.6 by simply replacing Concircular in part b of the theorem by projective, conformal, conharmonic, $H$–projective, $H$–conformal, $H$–conharmonic, $H$–Concircular, Conharmonic $C^*$, and Conformal$^*$. 

Now from 1.7.j we can have:

$$2.8) (D_vC^*)(X,Y,Z) - v(U)C^*(X,Y,Z) = (D_vR)(X,Y,Z)$$

$$- v(U)R(X,Y,Z) - \frac{1}{2(n-1)}[g(Y,Z)R(X,Y)]$$

$$- v(U)RX - g(X,Z)(D_vR)Y - v(U)RY]$$

$$+ \frac{(D_vr - v(U)r)}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y']$$

If the manifold is Conformal$^*$ recurrent we have;
\[ (D_u R)(X , Y , Z ) - v (U) R (X , Y , Z ) \]
\[
- \frac{1}{2(n - 1)} [g (Y , Z ) ((D_u R)X - v (U) RX )] \\
- g (X , Z ) ((D_u R)Y - v (U) RY ) \\
+ \frac{(D_u r - v (U) r)}{2(n - 1)(2n - 1)} [g (Y , Z )X - g (X , Z )Y ] = 0
\]

Contracting this equation with respect to \( X \) we get:

\[ 2.10 \]
\[
\frac{2(n - 1)}{2n - 2} ((D_u S)(Y , Z ) - v (U) S( Y , Z ) = 0
\]

Hence a Conformal* recurrent Kaehlar manifold is Ricci recurrent.

But from 1.17 and 1.7.j we have;

\[ 2.11 \]
\[
M (X , Y , Z ) = C * (X , Y , Z ) - \frac{1}{4(n - 1)} [S (Y , Z )X ] \\
- S (X , Z )Y + S (Y , Z )X ] - \frac{1}{2(n - 1)} [g (Y , Z )RX \\
- g (X , Z )RY ] + \frac{r}{2(n - 1)(2n - 1)} [g (Y , Z )X - g (X , Z )Y ]
\]

From which we can get;

\[ 2.12 \]
\[
(D_u M)(X , Y , Z ) - v (U) M (X , Y , Z ) = (D_u C *)(X , Y , Z ) \\
- v (U) C *(X , Y , Z ) - \frac{1}{4(n - 1)} [((D_u S)(Y , Z ) - v (U) S (Y , Z ))X \\
- ((D_u S)(X , Z ) - v (U) S (X , Z )Y )] - ((D_u S)(Y , Z ) - v (U) S (Y , Z ))Y ] \\
- ((D_u S)(X , Z ) - v (U) S (X , Z )Y ) ] - \frac{1}{2(n - 1)} [g (Y , Z )((D_u R)X \\
- g (X , Z )Y ].
\]

Therefore, we have in consequence of theorem 2.1 and equations 2.10 & 2.9.

**Theorem 2.7:** A Kaehler manifold \( M_{2n} , n > 2 \) is \( M \) – projective recurrent if and only if it is Conformal * recurrent.

Now from 1.7.d we have:

\[ 2.13 \]
\[
(D_v Y)(X , Y , Z ) - v (U) Y (X , Y , Z ) = (D_v R)(X , Y , Z ) \\
- v (U) R (X , Y , Z ) - \frac{(D_v r - v (U) r)}{2n(2n - 1)} [g (Y , Z )X - g (X , Z )Y ]
\]
If the manifold Concircular recurrent we have:

2.14) \((D_v R)(X, Y, Z) - v(U) R(X, Y, Z) = \frac{(D_v r - v(U)r)}{2n(2n - 1)} [g(Y, Z) X - g(X, Z) Y] \)

Contracting this equation with respect to \(X\) we obtain;

2.15) \((D_v S) Y, Z) - v(U) S(Y, Z) = \frac{(D_v r - v(U)r)}{2n} g(Y, Z) \)

If \(r = 0\), we have; \((D_v S) Y, Z) - v(U) S(Y, Z) = 0\), which means that the manifold is Ricci-recurrent. But from 1.17 and 1.7.d we can have:

2.16) \(M(X, Y, Z) = V(X, Y, Z) - \frac{1}{4(n - 1)} [S(Y, Z) X - S(X, Z) Y - S(Y, Z) X + S(X, Z) Y] + \frac{r}{2n(2n - 1)} [g(Y, Z) X - g(X, Z) Y] \)

Form which we can get:

2.17) \((D_v M)(X, Y, Z) - v(U) M(X, Y, Z) = (D_v V)(X, Y, Z) - v(U) V(X, Y, Z) \)

Hence, we have:

**Theorem 2.8:** A necessary and sufficient condition for a Kaehler manifold \(M_2, n > 2\) of zero scalar curvature to be \(M\)–projective recurrent is that it is Concircular recurrent manifold.

Similarly we can prove:

**Theorem 2.9:** A necessary and sufficient condition for a Kaehler manifold \(M_2, n > 2\) of zero scalar curvature to be \(M\)–projective recurrent is that it is \(H\)–Concircular manifold.

**Theorem 2.10:** A necessary and sufficient condition for a Kaehler manifold \(M_2, n > 2\) of zero for curvature to be \(M\)–projective recurrent is that it is Conharmonic* recurrent manifold.

### 3. M-projective symmetric Kaehler manifold

A Kaehler manifold is said to be \(M\)–projective symmetric if it satisfies.
3.1) \((D_uM)(X,Y,Z)=0\).

It is clear that every symmetric Kaehler manifold is \(M\)–projective symmetric.

From 1.17 and 3.1 we have if the manifold is \(M\)–projective symmetric,

\[ (D_uR)(X,Y,Z) - \frac{1}{4(n-1)}[(D_uS)(Y,Z)X - (D_uS)(X,Z)Y - (D_uS)(X,Z)Y^\perp] \]

\[ + (D_uS)(X,Z)Y^\perp = 0 \]

Contracting this equation with respect to \(X\) we get;

3.3) \(\frac{n-2}{2(n-1)}(D_uS)(Y,Z) = 0\).

If \(n \neq 2\), then the manifold is Ricci-Symmetric. That is, equations 1.13 and 1.14 holds. Hence, we have that,

**Theorem 3.1:** A necessary and sufficient condition for a \(M\)–projective symmetric Kaehler manifold to be symmetric is that it is Ricci-symmetric.

**Theorem 3.2:** An \(M\)–projective symmetric Kaehler manifold, \(M_{2n}\), \(n > 2\) is Ricci-symmetric.

**Theorem 3.3:** Every \(M\)–projective symmetric Kaehler manifold \(M_{2n}\), \(n > 2\) is symmetric.

**Theorem 3.4:** On \(M\)–projective symmetric Kaehler manifold \(M_{2n}\), \(n > 2\), the scalar curvature is constant.

Now using Bianchi identify on 3.2 we have:

3.4) \((D_uR)(Y,X,Z) - (D_uR)(U,X,Z) - \frac{1}{4(n-1)}[(D_uS)(Y,Z)X - (D_uS)(X,Z)Y]

\[ - (D_uS)(Y,Z)X + (D_uS)(X,Z)Y^\perp] = 0 \]

Contracting this equation with respect to \(U\) we get;

3.5) \(\frac{1}{4(n-1)}[(4n-5)(D_uS)(Y,Z) - (D_uS)(Y,Z)) + (D_{\chi}^{-1}H)(Y,Z) -

\[ (D_{\chi}^{-1}H)(X,Z))] = 0 \]

Hence we can state:

**Theorem 3.5:** On an \(M\)–projective symmetric Kaehler manifold we have equation 3.5.

**Theorem 3.6:** On an \(M\)–projective symmetric Kaehler manifold, the first covariant derivative of the Ricci tensor is symmetric if and only if

\((D_{\chi}^{-1}H)(Y,Z) = (D_{\chi}^{-1}H)(X,Z)\)

**Theorem 3.7:** Every Einstein \(M\)–Projective symmetric Kaehler manifold is symmetric.
M-projective curvature tensor on Kaehler manifold

**Proof:** For an Einstein manifold the scalar curvature is constant and the Ricci tensor is given by:

\[ S(X,Y) = \frac{r}{2n} g(X,Y). \]

Therefore, we have; \((D_v S)(X,Y) = 0\). Hence the statement follows from 3.2.

**Theorem 3.8:** Every recurrent \(M\) – projective symmetric Kaehler manifold is \(M\) – protectively flat.

The proof is obvious.

**Theorem 3.9:** An \(M\) – projective symmetric Kaehler manifold is Ricci-recurrent if and only if;

\[ (D_v R)(X,Y,Z) + v(U)[M(X,Y,Z) - R(X,Y,Z)] = 0 \]

**Proof:** If the manifold is Ricci recurrent then we have in consequence of 1.9 and 3.2.

\[ (D_v R)(X,Y,Z) = \frac{v(U)}{4(n-1)}[S(Y,Z)X - S(X,Z)Y - S(Y,Z)X + S(X,Z)Y] = 0 \]

Using 1.17 and 3.8 we get 3.7.

Conversely, if 3.7 is true then using it on 3.2 we get;

\[ (D_v S)(Y,Z) = v(U)S(Y,Z). \]

Hence we have the statement.

**Theorem 3.10:** A recurrent Einstein \(M\) – projective symmetric space is flat.

**Proof:** For an Einstein manifold we have:

\[ (D_v M)(X,Y,Z) = (D_v R)(X,Y,Z). \]

If the manifold is \(M\) – projective recurrent we have for a non-zero recurrence vector \(v\).

\[ (D_v M)(X,Y,Z) = v(U)M(X,Y,Z). \]

By theorem 2.9 we have;

\[ (D_v M)(X,Y,Z) = v(U)R(X,Y,Z). \]

Since the manifold is \(M\)-projective symmetric we have; \(v(U)R(X,Y,Z) = 0\). Hence we have the statement, since \(v\) non-zero.

Now differentiating 2.16 covariant we get;

\[ (D_u M)(X,Y,Z) = (D_u V)(X,Y,Z) - \frac{1}{4(n-1)}[(D_v S)(Y,Z)X - (D_v S)(X,Z)Y]
- (D_v S)(Y,Z)X - (D_v S)(X,Z)Y]
- \left[\frac{D_v r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)]\right]. \]

Therefore we can have in consequence of 1.13, 1.14, 1.16 and 3.1.

**Theorem 3.11:** On a Kaehler manifold if any two of the following hold, the third also hold.

**a.** It is \(M\) – projective symmetric manifold.

**b.** It is Concircular symmetric manifold.

**c.** It is Ricci symmetric manifold.
Similarly we can prove nine theorems analog to theorem 3.10 by simply replacing Concircular in part b of the theorem by projective, conformal, $H -$ conharmonic, $H -$ Concircular, Conharmonic*, and Conformal*.

**Theorem 3.12:** If an $M -$ projective symmetric Kaehler manifold is Concircular recurrent and Ricci- recurrent under the same recurrence Vector, then it is $M -$ protectively flat.

**Proof:** Using the facts given in theorem 2.10 we get;

\[
V(X, Y, Z) - \frac{1}{4(n-1)}[S(Y', Z)X - S(X, Z)Y' - S(Y', Z)X + S(X, Z)Y']
\]

\[
+ \frac{r}{2n(2n-1)}[g(Y', Z)X - g(X, Z)Y'] = 0
\]

Hence the result follows from 3.16.

**Theorem 3.13:** If a Kaehler manifold is Concircular symmetric, $M -$ projective recurrent and Ricci-recurrent under the same recurrent vector, then it is Concircular flat.

The proof is similar to the proof of the above theorem.

**Theorem 3.14:** If an $M -$ projective symmetric Kaehler manifold is Concircular symmetric and Ricci-recurrent under the same recurrent vector, then the $M -$ projective and the Concircular tensors coincide.

**Proof:** Using the fact given in theorem 3.10 we get:

\[
- \frac{1}{4(n-1)}[S(Y', Z)X - S(X, Z)Y' - S(Y', Z)X + S(X, Z)Y']
\]

\[
+ \frac{r}{2n(2n-1)}[h(Y', Z)X - g(X, Z)Y'] = 0
\]

Hence the statement follows from 3.9.

Similarly we can prove three theorem analog to the above three theorems by simply replacing Concircular in each one by projective, conformal, conharmonic*, $H -$ projective, $H -$ conformal, $H -$ Conharmonic, $H -$ Concircular, Conharmonic*, and Conformal*.

**References**


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