Stability of Quadratic Functional Equation in RN-Spaces

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Abstract

In this paper, using direct method, we prove the generalized Hyers-Ulam stability of the following quadratic functional equation

\[
\sum_{1 \leq i < j \leq m} f(x_i + x_j) + f(x_i - x_j) = 2(m - 1) \sum_{i=1}^{m} f(x_i)
\]

for all \(x_1, x_2, \ldots, x_m \in X\), where \(m \geq 2\) in random normed spaces.

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1 Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation \( D \) must be close to an exact solution of \( D \)?.

If the problem accepts a solution, we say that the equation \( D \) is stable. The first stability problem concerning group homomorphisms was raised by Ulam [8] in 1940.

We are given a group \( G \) and a metric group \( G' \) with metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if \( f : G \to G' \) satisfies \( d(f(xy), f(x)f(y)) < \delta \), for all \( x, y \in G \), then a homomorphism \( h : G \to G' \) exists with \( d(f(x), h(x)) < \varepsilon \) for all \( x \in G \)?.

Ulam's problem was partially solved by Hyers [4] in 1941. In 1978, Th. M. Rassias [5] formulated and proved the following theorem, which implies Hyers's Theorem as a special case. Suppose that \( E \) and \( F \) are real normed spaces with \( F \) a complete normed space, \( f : E \to F \) is a mapping such that for each fixed \( x \in E \) the mapping \( t \to f(tx) \) is continuous on \( R \), and let there exist \( \varepsilon > 0 \) and \( p \in [0, 1) \) such that for all \( x, y \in E \)

\[
\frac{||f(x + y) - f(x) - f(y)||}{||x||^p + ||y||^p} \leq \varepsilon
\]

Then there exists a unique linear mapping \( T : E \to F \) such that such that for all \( x \in E \)

\[
||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2^{p-1}}
\]

The functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [7] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. In [2], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

2 Preliminary Notes

In the sequel, we shall adopt the usual terminologies, notions and conventions of the theory of random normed spaces(see [6]). Throughout this paper, the space of all probability distribution functions is denoted by \( \Lambda^+ \). Elements of
$\Lambda^+$ are functions $F : R \cup [-\infty, +\infty] \to [0, 1]$, such that $F$ is left continuous and nondecreasing on $R$ and $F(0) = 0, F(+\infty) = 1$. It’s clear that the subset

$$D^+ = \{F \in \Lambda^+ : l^- F(-\infty) = 1\};$$

where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of $\Lambda^+$. The space $\Lambda^+$ is partially ordered by the usual point-wise ordering of functions, that is for all $t \in R$, $F \leq G$ if and only if $F(t) \leq G(t)$.

**Definition 2.1** A function $T : [0, 1]^2 \to [0, 1]$ is a continuous triangular norm (briefly a $t$-norm) if $T$ satisfies the following conditions:

(i) $T$ is commutative and associative;

(ii) $T$ is continuous;

(iii) $T(x, 1) = x$ for all $x \in [0, 1]$;

(iv) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0, 1]$.

**Definition 2.2** A random normed space (briefly RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:

(i) $\mu_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;

(ii) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in R, \alpha \neq 0, x \in X$ and $t \geq 0$.

(iii) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$, for all $x, y \in X$ and $t, s \geq 0$.

**Definition 2.3** Let $(X, \mu, T)$ be an RN-space. A sequence $\{x_n\}$ in $X$ is said to be converges to $x \in X$ if for all $t > 0$, $\lim_{n \to \infty} \mu_{x_n-x}(t) = 1$.

**Definition 2.4** A sequence $\{x_n\}$ in $(X, \mu, T)$ is said to be a Cauchy sequence in $X$ if for all $t > 0$, $\lim_{n \to \infty} \mu_{x_n-x_m}(t) = 1$. The RN-space $(X, \mu, T)$ is said to be complete if every Cauchy sequence in $X$ is convergent.

**Theorem 2.5** If $(X, \mu, T)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)$.

### 3 Main Results

**Lemma 3.1** Let $X$ and $Y$ be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation

$$\sum_{1 \leq i < j \leq m} f(x_i \pm x_j) = 2(m - 1) \sum_{i=1}^{m} f(x_i)$$

if and only if $f$ is quadratic.
**Proof:** Let \( f \) be a quadratic function. Assume the equation (4) is true for \( n \) by induction argument. By (3)

\[
f(x_i + x_{n+1}) + f(x_i - x_{n+1}) - 2f(x_i) - 2f(x_{n+1}) = 0
\]

for all \( i = 1, \ldots, n \). Adding up (3) and (5), we have the desired equation (3) for \( n + 1 \). Conversely, let \( f \) satisfy the equation (3). By letting \( x_i = 0 \) for all \( i = 1, 2, \ldots, n \), we have \( f(0) = 0 \). Replacing \( x_i = 0 \) for all \( i = 3, 4, \ldots, n \), we obtain the equation

\[
f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) = 0
\]

which implies that \( f \) is quadratic. The proof is complete.

**Theorem 3.2** Let \( X \) be a real linear space, \((Z, \mu', \min)\) be an RN-space, \( \phi : X^m \rightarrow Z \) be a function such that for some \( 0 < \alpha < 4 \),

\[
\mu'_\phi(2x_1, \ldots, 2x_m)(t) \geq \mu'_{\alpha\phi(x_1, \ldots, x_m)}(t) \quad \forall x_1, \ldots, x_n \in X, \ t > 0
\]

(7)

if \( f(0) = 0 \) and for all \( x_1, \ldots, x_m \in X \) and \( t > 0 \), \( \lim_{n \to \infty} \mu'_{\phi(2^n x_1, \ldots, 2^n x_n)}(4^n t) = 1 \). Let \((Y, \mu, \min)\) be a complete RN-space. If \( f : X \rightarrow Y \) is a mapping such that for all \( x_1, \ldots, x_n \in X \) and \( t > 0 \)

\[
\mu\sum_{1 \leq i < j \leq m} f(x_i + x_j) + f(x_i - x_j) - 2\sum_{i=1}^m f(x_i)(t) \geq \mu'_\phi(x_1, \ldots, x_m)(t),
\]

(8)

then the limit \( Q(x) := \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \) exist for all \( x \in X \) and \( Q : X \rightarrow Y \) is a unique quadratic mapping satisfying the inequality

\[
\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x, \ldots, x)}(\frac{m(m-1)(4-\alpha)t}{2}).
\]

(9)

**Proof:** Putting \( x_1 = x_2 = \cdots = x_m = x \) in (8), we obtain

\[
\mu\frac{m(m-1)}{2} f(2x) - 2m(m-1)f(x)(t) \geq \mu'_{\phi(x, \ldots, x)}(t).
\]

(10)

So

\[
\mu f(2x) - f(x)(t) \geq \mu'_{\phi(x, \ldots, x)}(2m(m-1)t).
\]

(11)

Replacing \( x \) by \( 2^n x \) in (11) and using (7), we obtain

\[
\mu f(2^n x) - f(x)(t) \geq \mu'_{\phi(2^n x, 2^n x, \ldots, 2^n x)}(2 \times 4^n m(m-1)t)
\]

\[
\geq \mu'_{\phi(x, \ldots, x)}(\frac{2 \times 4^n m(m-1)t}{\alpha^n}).
\]

(12)
So by (12), we obtain
\[
\mu_{f(2^{n+1}x)} - f(x) \left( \sum_{k=0}^{n-1} \frac{t \alpha^k}{2 \times 4^k m (m-1)} \right) = \mu \sum_{k=0}^{n-1} \frac{f(2^{k+1}x)}{2^{k+1}} \left( \sum_{k=0}^{n-1} \frac{2 \times 4^k m (m-1)}{t \alpha^k} \right) \geq T_{n=0}^{n-1} \left( \mu_{f(2^{k+1}x)} - \frac{f(2^{k+1}x)}{2^{k+1}} \left( \sum_{k=0}^{n-1} \frac{2 \times 4^k m (m-1)}{t \alpha^k} \right) \right) \geq T_{n=0}^{n-1} \left( \mu'_{\phi(x,x,\cdots,x)} (t) \right) = \mu'_{\phi(x,x,\cdots,x)} (t) \tag{13}
\]

This implies that
\[
\mu_{f(2^{n+1}x)} - f(x) (t) \geq \mu'_{\phi(x,x,\cdots,x)} \left( \sum_{k=0}^{n-1} \frac{t \alpha^k}{2 \times 4^k m (m-1)} \right) \tag{14}
\]

Replacing \( x \) by \( 2^p x \) in (14), we obtain
\[
\mu_{f(2^{p+n}x)} - \frac{f(2^{p}x)}{4^p} (t) \geq \mu'_{\phi(x,x,\cdots,x)} \left( \sum_{k=0}^{p+n-1} \frac{t \alpha^k}{2 \times 4^k m (m-1)} \right) \tag{15}
\]

As \( \lim_{p,n \to \infty} \mu'_{\phi(x,x,\cdots,x)} \left( \sum_{k=0}^{p+n-1} \frac{t \alpha^k}{2 \times 4^k m (m-1)} \right) = 1 \) then \( \left\{ \frac{f(2^{n}x)}{4^n} \right\}_{n=1}^{\infty} \) is a Cauchy sequence in complete RN-space \((Y, \mu, \min)\), so there exist some point \( C(x) \in Y \) such that \( \lim_{n \to \infty} \frac{f(2^{n}x)}{4^n} = Q(x) \). Fix \( x \in X \) and put \( p = 0 \) in (15). Then we obtain
\[
\mu_{f(2^{n}x)} - f(x) (t) \geq \mu'_{\phi(x,x,\cdots,x)} \left( \sum_{k=0}^{n-1} \frac{t \alpha^k}{2 \times 4^k m (m-1)} \right) \tag{16}
\]

and so, for every \( \epsilon > 0 \), we have
\[
\mu_{C(x)-f(x)} (t + \epsilon) \geq T \left( \mu_{Q(x)-\frac{f(2^n x)}{4^n}} (\epsilon), \mu_{f(2^n x)} - f(x) (t) \right) \geq T \left( \mu_{Q(x)-\frac{f(2^n x)}{4^n}} (\epsilon), \mu'_{\phi(x,x,\cdots,x)} \left( \sum_{k=0}^{n-1} \frac{t \alpha^k}{2 \times 4^k m (m-1)} \right) \right) \tag{17}
\]

Taking the limit as \( n \to \infty \), we get
\[
\mu_{C(x)-f(x)} (t + \epsilon) \geq \mu'_{\phi(x,x,\cdots,x)} \left( \frac{m(m-1)(4-\alpha)t}{2} \right) \tag{18}
\]

Since \( \epsilon \) was arbitrary by taking \( \epsilon \to 0 \) in (18), we obtain
\[
\mu_{C(x)-f(x)} (t) \geq \mu'_{\phi(x,x,\cdots,x)} \left( \frac{m(m-1)(4-\alpha)t}{2} \right) \tag{19}
\]

Replacing \( x \) and \( y \) by \( 2^nx \) and \( 2^ny \), respectively, in (8) and using this fact that \( \lim_{n \to \infty} \mu'_{\phi(2^n x_1,\cdots,2^n x_n)} (4^n t) = 1 \), we get for all \( x_1, \cdots, x_n \in X \) and for all \( t > 0 \)
\[
\sum_{1 \leq i < j \leq m} Q(x_i + x_j) + Q(x_i - x_j) = 2 \sum_{i=1}^{m} Q(x_i).
\]
Therefore, the mapping $Q$ is quadratic. To prove the uniqueness of mapping $Q$, assume that there exist another additive mapping $R : X \to Y$ which satisfies (9). Since for all $n \in \mathbb{N}$ and every $x \in X$, $Q(2^n x) = 4^n Q(x)$ and $R(2^n x) = 4^n R(x)$, we find that
\[
\mu_{Q(x)-R(x)}(t) = \lim_{n \to \infty} \mu_{Q(2^n x)-R(2^n x)}(t).
\] (20)

So
\[
\mu_{Q(2^n x)-R(2^n x)}(t) \geq \min\{\mu_{Q(2^n x)-R(2^n x)}(\frac{t}{2}), \mu_{Q(2^n x)-R(2^n x)}(\frac{t}{2})\} \\
\geq \mu'_{2^n x, \mu} m(m-1)4^n(4-\alpha) t \quad \frac{t}{2} \\
\geq \mu'_{2^n x, \mu} m(m-1)4^n(4-\alpha) t. 
\] (21)

Since $\lim_{p \to \infty} m(m-1)4^n(4-\alpha) = 0$, we get
\[
\lim_{p \to \infty} \mu'_{2^n x, \mu} m(m-1)4^n(4-\alpha) t = 1.
\]
Therefore, for all $t > 0$, $\mu_{Q(x)-R(x)}(t) = 1$ and so $Q(x) = R(x)$. This completes the proof.

**Corollary 3.3** Let $X$ be a real linear space, $(Z, \mu', \min)$ be an RN-space, and $(Y, \mu, \min)$ a complete RN-spaces. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ and for all $x_1, \ldots, x_m \in X$ and $t > 0$
\[
\mu_{\sum_{1 \leq i \leq j \leq m} f(x_i + x_j) + f(x_i - x_j) - 2 \sum_{i=1}^m f(x_i)}(t) \geq \mu'_{\sum_{i=1}^m \|x_i\|p} z_0(t), 
\] (22)
then there is a unique quadratic mapping $Q : X \to Y$ such that
\[
\mu_{f(x) - Q(x)}(t) \geq \mu'_{\|x\|p} z_0 \left( \frac{m(m-1)(4-4^p)t}{2m} \right). 
\] (23)

**Proof:** Let $\alpha = 4^p$ and $\phi : X^m \to Z$ defined by $\phi(x_1, \ldots, x_m) = (\sum_{i=1}^m \|x_i\|p} z_0$. Applying Theorem (3.2), we get desired result.

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