On the Prime Labeling of Generalized Petersen Graphs $P(n,3)$

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Abstract

A graph $G$ with vertex set $V$ is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge $xy$ in $E$, the labels assigned to $x$ and $y$ are relatively prime or coprime. A graph is called prime if it has a prime labeling. In this paper, we show that generalized Petersen graphs $P(n,3)$ are not prime for odd $n$, prime for even $n \leq 100$ and conjectured that $P(n,3)$ are prime for all even $n$.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$.

A graph $G$ is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge $xy$ in $E$, the labels assigned to $x$ and $y$ are relatively prime or coprime. A graph is called prime

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if it has a prime labeling. This concept was originated with Entringer and introduced by Tout, Dabboucy, and Howalla[10].

Roger Entringer conjectured that all trees are prime. Fu and Huang [2] proved that every tree with $n \leq 15$ vertices is prime. O’Pikhurko [6, 7] extended this result to all $n \leq 50$. Other prime graphs include all cycles and the disjoint union of $C_{2k}$ and $C_n$. Seoud, Diab, and Elsakhawi [8] showed that following graphs are prime: Fans, Helms, Flowers, Stars, $K_{2,n}$ and $K_{3,n}$ unless $n = 3$ or 7. They also showed that $P_n + \overline{K}_m (m \geq 3)$ is not prime. Kelli Carlson [1] proved that generalized Books and $C_m$-Snakes are prime graphs. Vilfred, Somasundaram, and Nicholas [11] have conjectured that the grid $P_m \times P_n$ is prime when $n$ is prime and $n > m$. This conjecture was proved by Sundaram, Ponraj, and Somasundaram [9]. In the same article they also showed that $P_n \times P_n$ is prime when $n$ is prime. The authors [4, 5] proved that the following graphs are prime: generalized Petersen graph $P(n,k)$ for even $n \leq 2500$ and not prime for odd $n$ and Knödel graphs $W_{3,n}$ for $n \leq 130$. We refer the readers to the dynamic survey by Gallian [3].

The generalized Petersen graphs $P(n,k)$ are defined to be a graph on $2n(n \geq 3)$ vertices with $V(P(n,k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 0 \leq i \leq n-1\}$, subscripts modulo $n$. In this paper, we show that generalized Petersen graphs $P(n,3)$ are not prime for odd $n$, prime for even $n \leq 100$, and conjecture that $P(n,3)$ are prime for all even $n$.

2 Main Results

Theorem 2.1. $P(n,3)$ is not prime for odd $n$.

Proof. By contradiction. Suppose that $P(n,3)$ is prime for some odd $n$, say $n_1$. Let $f$ be a prime labeling of $P(n_1,3)$. Then one of $\{f(v_0), f(v_1), \ldots, f(v_{n_1-1})\}$ and $\{f(u_0), f(u_1), \ldots, f(u_{n_1-1})\}$ must contains at least $\frac{n_1+1}{2}$ evens, i.e. there are at least two evens adjacent, a contradiction. \(\square\)

For even $n$, let

\[
\mathcal{N}_i = \{n : n + i \text{ is prime}\}, \quad \mathcal{N}_i^* = \{n : 2n + i \text{ is prime}\}, \\
\mathcal{N} = \bigcup_{-8 \leq i \leq 3} (\mathcal{N}_{2i+1} \cap \mathcal{N}_{2i+5} \cap \mathcal{N}_{2i+7}), \\
\mathcal{N}^* = \bigcup_{-5 \leq i \leq 6} (\mathcal{N}_{4i+1}^* \cap \mathcal{N}_{4i+5}^* \cap \mathcal{N}_{4i+7}^*).
\]

We will prove the following Theorem by Lemmas 2.4 - 2.27.

Theorem 2.2. $P(n,3)$ is prime for even $n \in \mathcal{N} \cup \mathcal{N}^*$.

Observation 2.3. $f(u)$ and $f(v)$ are coprime if they satisfy any one of the following conditions:
(1) \( f(u) = 1 \) or \( f(v) = 1 \),
(2) \( f(u) = 2 \) and \( f(v) \) is odd or \( f(v) = 2 \) and \( f(u) \) is odd,
(3) \( f(u) + f(v) \) is prime,
(4) \( |f(u) - f(v)| = 1 \),
(5) \( |f(u) - 2f(v)| = 1 \),
(6) \( |f(u) - 2f(v)| = p_1^{t_1}p_2^{t_2} \ldots p_k^{t_k} \) and \( f(v) \not\equiv 0 \pmod{p_i} \) \( (1 \leq i \leq k) \),
(7) \( |f(u) - f(v)| = p^t \) is a prime power and \( f(u) \not\equiv 0 \pmod{p} \).

**Lemma 2.4.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_3^* \cap \mathcal{N}_1^* \cap \mathcal{N}_3^* \).

**Proof.** We define the function \( f \) as follows:

Let

\[
\begin{align*}
  f(v_i) &= \begin{cases} 
    i + 2, & 0 \leq i \leq n - 4, \ i \mod 2 = 0, \\
    2n - i, & 0 \leq i \leq n - 4, \ i \mod 2 = 1, \\
    n - 1, & i = n - 3, \\
    n, & i = n - 2, \\
    n + 1, & i = n - 1,
  \end{cases} \\
  f(u_i) &= \begin{cases} 
    i + 1, & 0 \leq i \leq n - 4, \ i \mod 2 = 0, \\
    2n - i - 1, & 0 \leq i \leq n - 4, \ i \mod 2 = 1, \\
    n + 2, & i = n - 3, \\
    n + 3, & i = n - 2, \\
    2n, & i = n - 1.
  \end{cases}
\end{align*}
\]

![Figure 1: \( P(n, 3) \) for \( n = 8 \in \mathcal{N}_3^* \cap \mathcal{N}_1^* \cap \mathcal{N}_3^* \).](image)

In Figure 2.1, we show the prime labeling of \( P(n, 3) \), where \( n = 8 \in \mathcal{N}_3^* \cap \mathcal{N}_1^* \cap \mathcal{N}_3^* \).

For \( 0 \leq i \leq n - 5 \), by Observation 2.3(3), \( f(v_i) \) and \( f(v_{i+1}) \) are coprime.

For \( n - 4 \leq i \leq n - 2 \), \( |f(v_{i+1}) - f(v_i)| = 1 \), by Observation 2.3(4), \( f(v_i) \) and \( f(v_{i+1}) \) are coprime. For \( i = n - 1 \), \( f(v_0) = 2 \) and \( f(v_{n-1}) = n + 1 \) is odd, by Observation 2.2(2), \( f(v_{n-1}) \) and \( f(v_0) \) are coprime.

For \( 0 \leq i \leq n - 7 \), by Observation 2.3(3), \( f(u_i) \) and \( f(u_{i+3}) \) are coprime.

For \( i = n - 6 \), \( |f(u_{n-6}) - f(u_{n-3})| = |n - 5 - (n + 2)| = 7 \), by Observation 2.3(7), \( f(u_{n-6}) \) and \( f(u_{n-3}) \) are coprime. For \( i = n - 5 \), \( |f(u_{n-5}) - f(u_{n-2})| = |n + 4 - (n + 3)| = 1 \), by Observation 2.3(4), \( f(u_{n-5}) \) and \( f(u_{n-2}) \) are coprime.

For \( i = n - 4 \), \( |f(u_{n-1}) - f(u_{n-4})| = |2n - 2(n - 3)| = 2 \times 3 \). Since \( n \in \mathcal{N}_3^* \cap \mathcal{N}_1^* \cap \mathcal{N}_3^* \), \( n - 3 \mod 3 \neq 0 \). Since \( n - 3 \) is odd, we have \( f(u_{n-4}) \) and \( f(u_{n-1}) \) are coprime. For \( i = n - 3 \), by Observation 2.3(1), \( f(u_{n-3}) = n + 2 \) and
$f(u_0) = 1$ are coprime. For $i = n - 2$, $|f(u_{n-2}) - f(u_{n-7})| = |n + 3 - (n + 6)| = 3$, by Observation 2.3(7), $f(u_{n-2})$ and $f(u_{n-7})$ are coprime. For $i = n - 1$, $f(u_{n-6}) = 3$, $f(u_{n-1}) = 2n$. Since $n \in \mathcal{N}_{-3}^* \cap \mathcal{N}_1^* \cap \mathcal{N}_3^*$, $2n \mod 3 \neq 0$, hence $f(u_{n-1})$ and $f(u_{n-6})$ are coprime.

For $0 \leq i \leq 4$,

$$|f(u_i) - f(v_i)| = \begin{cases} i + 1 - (i + 2) = 1, & i \mod 2 = 0, \\ 2n - i - 1 - (2n - i) = 1, & i \mod 2 = 1, \end{cases}$$

by Observation 2.3(4), $f(v_i)$ and $f(u_i)$ are coprime. For $i = n - 3$, $|f(u_{n-3}) - f(v_{n-3})| = |n + 2 - (n - 1)| = 3$, by Observation 2.3(7), $f(v_{n-3})$ and $f(u_{n-3})$ are coprime. For $i = n - 2$, $|f(u_{n-2}) - f(v_{n-2})| = |n + 3 - n| = 3$, by Observation 2.3(7), $f(v_{n-3})$ and $f(u_{n-3})$ are coprime. For $i = n - 1$, $|f(u_{n-1}) - 2f(v_{n-1})| = |2n - 2(n + 1)| = 2$ and $f(v_{n-1})$ is odd, by Observation 2.3(6) $f(v_{n-1})$, $f(u_{n-1})$ are coprime.

Hence $f$ is a prime labeling of $P(n, 3)$ for even $n \in \mathcal{N}_{-3}^* \cap \mathcal{N}_1^* \cap \mathcal{N}_3^*$. \hfill $\Box$

For the Lemmas 2.5 - 2.27, we only define $f$, and leave for the readers to verify that the $f$ is a prime labeling of $P(n, 3)$.

**Lemma 2.5.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-7}^* \cap \mathcal{N}_{-3}^* \cap \mathcal{N}_{-1}^*$.

**Proof.** We define the function $f$ as follows:

Let

$$f(v_i) = \begin{cases} i + 2, & 0 \leq i \leq n - 4, i \mod 2 = 0, \\ 2n - i - 4, & 0 \leq i \leq n - 4, i \mod 2 = 1, \\ 2n - 3, & i = n - 3, \\ 2n - 4, & i = n - 2, \\ 2n - 1, & i = n - 1, \end{cases} \quad f(u_i) = \begin{cases} i + 1, & 0 \leq i \leq n - 4, i \mod 2 = 0, \\ 2n - i - 5, & 0 \leq i \leq n - 4, i \mod 2 = 1, \\ 2n - 2, & i = n - 3, \\ n - 1, & i = n - 2, \\ 2n, & i = n - 1. \end{cases}$$

In Figure 2.2(a), we show the prime labeling of $P(n, 3)$, where $n = 10 \in \mathcal{N}_{-7}^* \cap \mathcal{N}_{-3}^* \cap \mathcal{N}_{-1}^*$.

**Lemma 2.6.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-11}^* \cap \mathcal{N}_{-7}^* \cap \mathcal{N}_{-5}^*$.

**Proof.** We define the function $f$ as follows:

Let

$$f(v_i) = \begin{cases} i + 2, & 0 \leq i \leq n - 5, i \mod 2 = 0, \\ 2n - i - 8, & 0 \leq i \leq n - 5, i \mod 2 = 1, \\ 2n - 4, & i = n - 4, \\ 2n - 7, & i = n - 3, \\ 2n, & i = n - 2, \\ 2n - 1, & i = n - 1, \end{cases} \quad f(u_i) = \begin{cases} i + 1, & 0 \leq i \leq n - 6, i \mod 2 = 0, \\ 2n - i - 9, & 1 \leq i \leq n - 6, i \mod 2 = 1, \\ 2n - 8, & i = n - 5, \\ 2n - 3, & i = n - 4, \\ 2n - 6, & i = n - 3, \\ 2n - 5, & i = n - 2, \\ 2n - 2, & i = n - 1. \end{cases}$$

In Figure 2.2(b), we show the prime labeling of $P(n, 3)$, where $n = 12 \in \mathcal{N}_{-11}^* \cap \mathcal{N}_{-7}^* \cap \mathcal{N}_{-5}^*$.

**Lemma 2.7.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-15}^* \cap \mathcal{N}_{-11}^* \cap \mathcal{N}_{-9}^*$.

**Proof.** We define the function $f$ as follows:
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Let

\[
f(v_i) = \begin{cases} 
    i + 2, & 0 \leq i \leq n - 7, i \mod 2 = 0, \\
    2n - i - 12, & 0 \leq i \leq n - 7, i \mod 2 = 1, \\
    2n - 8, & i = n - 6, \\
    2n - 9, & i = n - 5, \\
    2n - 4, & i = n - 4, \\
    2n - 3, & i = n - 3, \\
    2n - 2, & i = n - 2, \\
    2n - 1, & i = n - 1,
\end{cases}
\]

\[
f(u_i) = \begin{cases} 
    i + 1, & 0 \leq i \leq n - 8, i \mod 2 = 0, \\
    2n - i - 13, & 0 \leq i \leq n - 8, i \mod 2 = 1, \\
    2n - 12, & i = n - 7, \\
    2n - 7, & i = n - 6, \\
    2n - 10, & i = n - 5, \\
    2n - 11, & i = n - 4, \\
    2n - 6, & i = n - 3, \\
    2n - 5, & i = n - 2, \\
    2n, & i = n - 1.
\end{cases}
\]

In Figure 2.3(a), we show the prime labeling of $P(n,3)$, where $n = 14 \in \mathcal{N}_{-15}^* \cap \mathcal{N}_{-11}^* \cap \mathcal{N}_{-9}^*$.

\[P(10,3), 10 \in \mathcal{N}_{-7}^* \cap \mathcal{N}_{-3}^* \cap \mathcal{N}_{-1}_{1}^* \]

\[P(12,3), 12 \in \mathcal{N}_{-11}^* \cap \mathcal{N}_{-7}^* \cap \mathcal{N}_{-5}^* \]

\[P(14,3), 14 \in \mathcal{N}_{-15}^* \cap \mathcal{N}_{-11}^* \cap \mathcal{N}_{-9}^* \]

\[P(16,3), 16 \in \mathcal{N}_{-19}^* \cap \mathcal{N}_{-15}^* \cap \mathcal{N}_{-13}^* \]

**Lemma 2.8.** $P(n,3)$ is prime for even $n \in \mathcal{N}_{-19}^* \cap \mathcal{N}_{-15}^* \cap \mathcal{N}_{-13}^*$.

**Proof.** We define the function $f$ as follows:
Let

\[
\begin{align*}
\text{f(v)} = & \begin{cases} 
 i + 2, & 0 \leq i \leq n - 9, i \mod 2 = 0, \\
2n - i - 16, & 0 \leq i \leq n - 9, i \mod 2 = 1, \\
2n - 12, & i = n - 8, \\
2n - 13, & i = n - 7, \\
2n - 8, & i = n - 6, \\
2n - 1, & i = n - 5, \\
2n - 2, & i = n - 4, \\
2n - 3, & i = n - 3, \\
2n - 4, & i = n - 2, \\
2n - 5, & i = n - 1,
\end{cases} \\
\text{f(u)} = & \begin{cases} 
 i + 1, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
2n - i - 17, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
2n - 16, & i = n - 9, \\
2n - 11, & i = n - 8, \\
2n - 14, & i = n - 7, \\
2n - 7, & i = n - 6, \\
2n - 10, & i = n - 5, \\
2n - 15, & i = n - 4, \\
2n - 6, & i = n - 3, \\
2n - 9, & i = n - 2, \\
2n, & i = n - 1.
\end{cases}
\end{align*}
\]

In Figure 2.3(b), we show the prime labeling of \( P(n, 3) \), where \( n = 16 \in \mathcal{N}^*_1 \cap \mathcal{N}^*_5 \cap \mathcal{N}^*_7 \).

**Lemma 2.9.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}^*_1 \cap \mathcal{N}^*_5 \cap \mathcal{N}^*_7 \).

**Proof.** We define the function \( f \) as follows:

Let

\[
\begin{align*}
\text{f(v)} = & \begin{cases} 
 6 + i, & 0 \leq i \leq n - 3, i \mod 2 = 0, \\
2n - i, & 0 \leq i \leq n - 3, i \mod 2 = 1, \\
4, & i = n - 2, \\
1, & i = n - 1,
\end{cases} \\
\text{f(u)} = & \begin{cases} 
 5 + i, & 0 \leq i \leq n - 4, i \mod 2 = 0, \\
2n - i - 1, & 0 \leq i \leq n - 4, i \mod 2 = 1, \\
2, & i = n - 3, \\
3, & i = n - 2, \\
2n, & i = n - 1.
\end{cases}
\end{align*}
\]

In Figure 2.4(a), we show the prime labeling of \( P(n, 3) \), where \( n = 18 \in \mathcal{N}^*_1 \cap \mathcal{N}^*_5 \cap \mathcal{N}^*_7 \).

\[ \text{P(18, 3), 18} \in \mathcal{N}^*_1 \cap \mathcal{N}^*_5 \cap \mathcal{N}^*_7 \]

\[ \text{P(16, 3), 16} \in \mathcal{N}^*_5 \cap \mathcal{N}^*_9 \cap \mathcal{N}^*_11 \]

**Figure 4:**

**Lemma 2.10.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}^*_5 \cap \mathcal{N}^*_9 \cap \mathcal{N}^*_11 \).

**Proof.**

In \( \mathcal{N}^*_5 \cap \mathcal{N}^*_9 \cap \mathcal{N}^*_11 \), there is only one integer smaller than 16, namely 4. Since \( 4 \in \mathcal{N}^*_7 \cap \mathcal{N}^*_3 \cap \mathcal{N}^*_1 \), by Lemma 2.5, \( P(4, 3) \) is prime. Hence, we only consider even \( n \geq 16 \). And we define the function \( f \) as follows:
Let

\[
\begin{align*}
    f(v_i) &= \begin{cases} 
        10 + i, & 0 \leq i \leq n - 5, \ i \ mod \ 2 = 0, \\
        2n - i, & 0 \leq i \leq n - 5, \ i \ mod \ 2 = 1, \\
        4, & i = n - 4, \\
        5, & i = n - 3, \\
        6, & i = n - 2, \\
        1, & i = n - 1,
    \end{cases} \\
    f(u_i) &= \begin{cases} 
        9 + i, & 0 \leq i \leq n - 6, \ i \ mod \ 2 = 0, \\
        2n - i - 1, & 0 \leq i \leq n - 6, \ i \ mod \ 2 = 1, \\
        2, & i = n - 5, \\
        3, & i = n - 4, \\
        8, & i = n - 3, \\
        7, & i = n - 2, \\
        2n, & i = n - 1.
    \end{cases}
\end{align*}
\]

In Figure 2.4(b), we show the prime labeling of \( P(n, 3) \), where \( n = 16 \in \mathcal{N}_5^* \cap \mathcal{N}_9^* \cap \mathcal{N}_{11}^* \).

**Lemma 2.11.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_9^* \cap \mathcal{N}_{13}^* \cap \mathcal{N}_{15}^* \).

**Proof.** We define the function \( f \) as follows:

Let

\[
\begin{align*}
    f(v_i) &= \begin{cases} 
        14 + i, & 0 \leq i \leq n - 7, \ i \ mod \ 2 = 0, \\
        2n - i, & 0 \leq i \leq n - 7, \ i \ mod \ 2 = 1, \\
        4, & i = n - 6, \\
        3, & i = n - 5, \\
        10, & i = n - 4, \\
        7, & i = n - 3, \\
        12, & i = n - 2, \\
        5, & i = n - 1,
    \end{cases} \\
    f(u_i) &= \begin{cases} 
        13 + i, & 0 \leq i \leq n - 8, \ i \ mod \ 2 = 0, \\
        2n - i - 1, & 0 \leq i \leq n - 8, \ i \ mod \ 2 = 1, \\
        2, & i = n - 7, \\
        1, & i = n - 6, \\
        8, & i = n - 5, \\
        9, & i = n - 4, \\
        6, & i = n - 3, \\
        11, & i = n - 2, \\
        2n, & i = n - 1.
    \end{cases}
\end{align*}
\]

In Figure 2.5(a), we show the prime labeling of \( P(n, 3) \), where \( n = 14 \in \mathcal{N}_9^* \cap \mathcal{N}_{13}^* \cap \mathcal{N}_{15}^* \).

**Figure 5:**

**Lemma 2.12.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_{13}^* \cap \mathcal{N}_{17}^* \cap \mathcal{N}_{19}^* \).

**Proof.** We define the function \( f \) as follows:
Let

\[ f(v_i) = \begin{cases} 
18 + i, & 0 \leq i \leq n - 9, \text{mod} 2 = 0, \\
2n - i, & 0 \leq i \leq n - 9, \text{mod} 2 = 1, \\
4, & i = n - 8, \\
5, & i = n - 7, \\
8, & i = n - 6, \\
9, & i = n - 5, \\
10, & i = n - 4, \\
13, & i = n - 3, \\
16, & i = n - 2, \\
11, & i = n - 1, 
\end{cases} \quad f(u_i) = \begin{cases} 
17 + i, & 0 \leq i \leq n - 10, \text{mod} 2 = 0, \\
2n - i, & 0 \leq i \leq n - 10, \text{mod} 2 = 1, \\
2, & i = n - 9, \\
3, & i = n - 8, \\
6, & i = n - 7, \\
1, & i = n - 6, \\
14, & i = n - 5, \\
7, & i = n - 4, \\
2n, & i = n - 3, \\
15, & i = n - 2, \\
12, & i = n - 1. 
\end{cases} \]

In Figure 2.5(b), we show the prime labeling of \( P(n, 3) \), where \( n = 12 \in \mathcal{N}_1^* \cap \mathcal{N}_7^* \cap \mathcal{N}_9^* \).

**Lemma 2.13.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_1^* \cap \mathcal{N}_7^* \cap \mathcal{N}_9^* \).

**Proof.**

In \( \mathcal{N}_7^* \cap \mathcal{N}_9^* \cap \mathcal{N}_9^* \), there is only one integer smaller than 40, namely 10. Since \( 10 \in \mathcal{N}_7^* \cap \mathcal{N}_9^* \cap \mathcal{N}_9^* \), by Lemma 2.5, \( P(10, 3) \) is prime. Hence, we only consider even \( n \geq 40 \). And we define the function \( f \) as follows:

Let

\[ f(v_i) = \begin{cases} 
22 + i, & 0 \leq i \leq n - 11, \text{mod} 2 = 0, \\
2n - i, & 0 \leq i \leq n - 11, \text{mod} 2 = 1, \\
4, & i = n - 10, \\
3, & i = n - 9, \\
10, & i = n - 8, \\
9, & i = n - 7, \\
14, & i = n - 6, \\
5, & i = n - 5, \\
12, & i = n - 4, \\
11, & i = n - 3, \\
18, & i = n - 2, \\
13, & i = n - 1, 
\end{cases} \quad f(u_i) = \begin{cases} 
21 + i, & 0 \leq i \leq n - 12, \text{mod} 2 = 0, \\
2n - i, & 0 \leq i \leq n - 12, \text{mod} 2 = 1, \\
2, & i = n - 10, \\
1, & i = n - 9, \\
8, & i = n - 7, \\
7, & i = n - 6, \\
15, & i = n - 5, \\
17, & i = n - 4, \\
16, & i = n - 3, \\
19, & i = n - 2, \\
20, & i = n - 1. 
\end{cases} \]

**Lemma 2.14.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_2^* \cap \mathcal{N}_5^* \cap \mathcal{N}_7^* \).

**Proof.**

In \( \mathcal{N}_2^* \cap \mathcal{N}_5^* \cap \mathcal{N}_7^* \), there is only one integer smaller than 38, namely 8. Since \( 8 \in \mathcal{N}_2^* \cap \mathcal{N}_5^* \cap \mathcal{N}_7^* \), by Lemma 2.4, \( P(8, 3) \) is prime. Hence, we only consider even \( n \geq 38 \). And we define the function \( f \) as follows:

Let

\[ f(v_i) = \begin{cases} 
26 + i, & 0 \leq i \leq n - 13, \text{mod} 2 = 0, \\
2n - i, & 0 \leq i \leq n - 13, \text{mod} 2 = 1, \\
4, & i = n - 12, \\
3, & i = n - 11, \\
10, & i = n - 10, \\
9, & i = n - 9, \\
16, & i = n - 8, \\
11, & i = n - 7, \\
12, & i = n - 6, \\
17, & i = n - 5, \\
18, & i = n - 4, \\
19, & i = n - 3, \\
20, & i = n - 2, \\
21, & i = n - 1, 
\end{cases} \quad f(u_i) = \begin{cases} 
25 + i, & 0 \leq i \leq n - 14, \text{mod} 2 = 0, \\
2n - i, & 0 \leq i \leq n - 14, \text{mod} 2 = 1, \\
2, & i = n - 13, \\
1, & i = n - 12, \\
8, & i = n - 11, \\
7, & i = n - 10, \\
14, & i = n - 9, \\
15, & i = n - 8, \\
6, & i = n - 7, \\
13, & i = n - 6, \\
22, & i = n - 5, \\
5, & i = n - 4, \\
24, & i = n - 3, \\
23, & i = n - 2, \\
2n, & i = n - 1. 
\end{cases} \]
Lemma 2.15. \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_{25}^* \cap \mathcal{N}_{29}^* \cap \mathcal{N}_{31}^* \).

Proof. In \( \mathcal{N}_{25}^* \cap \mathcal{N}_{29}^* \cap \mathcal{N}_{31}^* \), there is only one integer smaller than 36, namely 6. Since \( 6 \in \mathcal{N}_{25}^* \cap \mathcal{N}_{29}^* \cap \mathcal{N}_{31}^* \), by Lemma 2.9, \( P(6, 3) \) is prime. Hence, we only consider even \( n \geq 36 \). And we define the function \( f \) as follows:

\[
\begin{align*}
f(v_i) &= \begin{cases} 
30 + i, & 0 \leq i \leq n - 15, i \mod 2 = 0, \\
2n - i, & 0 \leq i \leq n - 15, i \mod 2 = 1, \\
i, & i = n - 14, \\
i, & i = n - 13, \\
i, & i = n - 12, \\
i, & i = n - 11, \\
i, & i = n - 10, \\
i, & i = n - 9, \\
i, & i = n - 8, \\
i, & i = n - 7, \\
i, & i = n - 6, \\
i, & i = n - 5, \\
i, & i = n - 4, \\
i, & i = n - 3, \\
i, & i = n - 2, \\
i, & i = n - 1, \\
\end{cases} \\
f(u_i) &= \begin{cases} 
29 + i, & 0 \leq i \leq n - 16, i \mod 2 = 0, \\
2n - i - 1, & 0 \leq i \leq n - 16, i \mod 2 = 1, \\
i, & i = n - 15, \\
i, & i = n - 14, \\
i, & i = n - 13, \\
i, & i = n - 12, \\
i, & i = n - 11, \\
i, & i = n - 10, \\
i, & i = n - 9, \\
i, & i = n - 8, \\
i, & i = n - 7, \\
i, & i = n - 6, \\
i, & i = n - 5, \\
i, & i = n - 4, \\
i, & i = n - 3, \\
i, & i = n - 2, \\
i, & i = n - 1, \\
\end{cases}
\end{align*}
\]

In Figure 2.6(a) we show the prime labeling of \( P(n, 3) \) for even \( n = 36 \in \mathcal{N}_{25}^* \cap \mathcal{N}_{29}^* \cap \mathcal{N}_{31}^* \).

![Figure 2.6(a)](image)

Lemma 2.16. \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_{-3} \cap \mathcal{N}_{1} \cap \mathcal{N}_{3} \).

Proof. We define the function \( f \) as follows:

\[
\begin{align*}
f(v_i) &= \begin{cases} 
i + 1, & 0 \leq i \leq n - 4, i \mod 2 = 0, \\
n + i + 3, & 0 \leq i \leq n - 4, i \mod 2 = 1, \\
n + 2, & i = n - 3, \\
n + 2, & i = n - 2, \\
2n - 1, & i = n - 1, \\
2n, & i = n - 1, \\
\end{cases} \\
f(u_i) &= \begin{cases} 
i + 2, & 0 \leq i \leq n - 4, i \mod 2 = 0, \\
n + i + 2, & 0 \leq i \leq n - 4, i \mod 2 = 1, \\
n + 1, & i = n - 3, \\
n, & i = n - 2, \\
n - 1, & i = n - 1. \\
\end{cases}
\end{align*}
\]
In Figure 2.6(b) we show the prime labeling of $P(n, 3)$ for even $n = 10 \in \mathcal{N}_{-3} \cap \mathcal{N}_1 \cap \mathcal{N}_3$.

**Lemma 2.17.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1} \cap \mathcal{N}_1$.

*Proof.* We define the function $f$ as follows:

Let

$$f(v_i) = \begin{cases} i + 1, & 0 \leq i \leq n - 3, \mod 2 = 0, \\ n + i + 1, & 0 \leq i \leq n - 3, \mod 2 = 1, \\ 2n - 1, & i = n - 2, \\ n, & i = n - 1, \end{cases} \quad f(u_i) = \begin{cases} i + 2, & 0 \leq i \leq n - 3, \mod 2 = 0, \\ n + i, & 0 \leq i \leq n - 3, \mod 2 = 1, \\ 2n, & i = n - 2, \\ n - 1, & i = n - 1. \end{cases}$$

In Figure 2.7(a), we show the prime labeling of $P(n, 3)$, where $n = 12 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1} \cap \mathcal{N}_1$.

![Figure 7](image_url)

**Lemma 2.18.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-7} \cap \mathcal{N}_{-3} \cap \mathcal{N}_{-1}$.

*Proof.* We define the function $f$ as follows:

Let

$$f(v_i) = \begin{cases} n + i - 2, & 0 \leq i \leq n - 4, \mod 2 = 0, \\ i, & 0 \leq i \leq n - 4, \mod 2 = 1, \\ 2n - 1, & i = n - 3, \\ 2n - 2, & i = n - 2, \\ 2n - 5, & i = n - 1, \end{cases} \quad f(u_i) = \begin{cases} n + i - 3, & 0 \leq i \leq n - 5, \mod 2 = 0, \\ i + 1, & 0 \leq i \leq n - 5, \mod 2 = 1, \\ 2n - 3, & i = n - 4, \\ 2n, & i = n - 3, \\ 2n - 7, & i = n - 2, \\ 2n - 4, & i = n - 1. \end{cases}$$

In Figure 2.7(b), we show the prime labeling of $P(n, 3)$, where $n = 8 \in \mathcal{N}_{-7} \cap \mathcal{N}_{-3} \cap \mathcal{N}_{-1}$.

**Lemma 2.19.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5} \cap \mathcal{N}_{-3}$.

*Proof.*

In $\mathcal{N}_{-9} \cap \mathcal{N}_{-5} \cap \mathcal{N}_{-3}$, there is only one integer smaller than 22, namely 16. Since $16 \in \mathcal{N}_{-3} \cap \mathcal{N}_1 \cap \mathcal{N}_3$, by Lemma 2.16, $P(16, 3)$ is prime. Hence, we only consider even $n \geq 22$. And we define the function $f$ as follows:
Let

\[ f(v_i) = \begin{cases} n + i - 4, & 0 \leq i \leq n - 6, \mod \equiv 0, \\
i, & 0 \leq i \leq n - 6, \mod \equiv 1, \\
2n - 5, & i = n - 5, \\
2n, & i = n - 4, \\
2n - 3, & i = n - 3, \\
2n - 4, & i = n - 2, \\
2n - 7, & i = n - 1, \end{cases} \]

\[ f(u_i) = \begin{cases} n + i - 5, & 0 \leq i \leq n - 7, \mod \equiv 0, \\
i + 1, & 0 \leq i \leq n - 7, \mod \equiv 1, \\
2n - 1, & i = n - 6, \\
2n - 6, & i = n - 5, \\
2n - 9, & i = n - 4, \\
2n - 2, & i = n - 3, \\
2n - 11, & i = n - 2, \\
2n - 8, & i = n - 1. \end{cases} \]

In Figure 2.8(a), we show the prime labeling of \( P(n, 3) \), where \( n = 22 \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5} \cap \mathcal{N}_{-3} \).

**Lemma 2.20.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7} \cap \mathcal{N}_{-5} \).

**Proof.** We define the function \( f \) as follows:

Let

\[ f(v_i) = \begin{cases} n + i - 6, & 0 \leq i \leq n - 8, \mod \equiv 0, \\
i, & 0 \leq i \leq n - 8, \mod \equiv 1, \\
2n - 9, & i = n - 7, \\
2n - 4, & i = n - 6, \\
2n - 3, & i = n - 5, \\
2n - 2, & i = n - 4, \\
2n - 11, & i = n - 3, \\
2n, & i = n - 2, \\
2n - 7, & i = n - 1, \end{cases} \]

\[ f(u_i) = \begin{cases} n + i - 7, & 0 \leq i \leq n - 8, \mod \equiv 0, \\
i + 1, & 0 \leq i \leq n - 8, \mod \equiv 1, \\
2n - 8, & i = n - 7, \\
2n - 13, & i = n - 6, \\
2n - 10, & i = n - 5, \\
2n - 5, & i = n - 4, \\
2n - 12, & i = n - 3, \\
2n - 1, & i = n - 2, \\
2n - 6, & i = n - 1. \end{cases} \]

In Figure 2.8(b), we show the prime labeling of \( P(n, 3) \), where \( n = 18 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7} \cap \mathcal{N}_{-5} \).

**Lemma 2.21.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_{-13} \cap \mathcal{N}_{-9} \cap \mathcal{N}_{-7} \).

**Proof.**

In \( \mathcal{N}_{-13} \cap \mathcal{N}_{-9} \cap \mathcal{N}_{-7} \), there is only one integer smaller than 26, namely 20. Since 20 \( \in \mathcal{N}_{-7} \cap \mathcal{N}_{-3} \cap \mathcal{N}_{-1} \), by Lemma 2.18, \( P(20, 3) \) is prime. Hence, we only consider even \( n \geq 26 \). And we define the function \( f \) as follows:

**Case 1.** \( n \equiv 1 \mod 7 \). Let
In Figure 2.9(a), we show the prime labeling of $P(n,3)$, where $n = 26 \in \mathcal{N}_{-13} \cap \mathcal{N}_{-9} \cap \mathcal{N}_{-7}$. 

Case 2. $n \equiv 3 \mod 7$. Let

$$f(v_i) = \begin{cases} 
  n + i - 8, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
  i, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
  2n - 17, & i = n - 9, \\
  2n - 14, & i = n - 8, \\
  2n - 9, & i = n - 7, \\
  2n - 10, & i = n - 6, \\
  2n - 3, & i = n - 5, \\
  2n - 8, & i = n - 4, \\
  2n - 1, & i = n - 3, \\
  2n - 6, & i = n - 2, \\
  2n - 15, & i = n - 1, 
\end{cases}$$

$$f(u_i) = \begin{cases} 
  n + i - 9, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
  i + 1, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
  2n - 16, & i = n - 9, \\
  2n - 13, & i = n - 8, \\
  2n - 12, & i = n - 7, \\
  2n - 11, & i = n - 6, \\
  2n, & i = n - 5, \\
  2n - 5, & i = n - 4, \\
  2n - 2, & i = n - 3, \\
  2n - 7, & i = n - 2, \\
  2n - 4, & i = n - 1. 
\end{cases}$$

Case 3. $n \equiv 4 \mod 7$. Let

$$f(v_i) = \begin{cases} 
  n + i - 8, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
  i, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
  2n - 17, & i = n - 9, \\
  2n - 14, & i = n - 8, \\
  2n - 9, & i = n - 7, \\
  2n - 15, & i = n - 6, \\
  2n - 6, & i = n - 5, \\
  2n - 2, & i = n - 4, \\
  2n - 1, & i = n - 3, \\
  2n - 10, & i = n - 2, \\
  2n - 9, & i = n - 1, 
\end{cases}$$

$$f(u_i) = \begin{cases} 
  n + i - 9, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
  i + 1, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
  2n - 16, & i = n - 9, \\
  2n - 13, & i = n - 8, \\
  2n - 12, & i = n - 7, \\
  2n - 11, & i = n - 6, \\
  2n, & i = n - 5, \\
  2n - 5, & i = n - 4, \\
  2n - 2, & i = n - 3, \\
  2n - 7, & i = n - 2, \\
  2n - 4, & i = n - 1. 
\end{cases}$$

Case 4. $n \not\equiv 1,3,4 \mod 7$. Let

$$f(v_i) = \begin{cases} 
  n + i - 8, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
  i, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
  2n - 17, & i = n - 9, \\
  2n - 14, & i = n - 8, \\
  2n - 9, & i = n - 7, \\
  2n - 15, & i = n - 6, \\
  2n - 6, & i = n - 5, \\
  2n - 2, & i = n - 4, \\
  2n - 12, & i = n - 3, \\
  2n - 3, & i = n - 2, \\
  2n - 11, & i = n - 1, 
\end{cases}$$

$$f(u_i) = \begin{cases} 
  n + i - 9, & 0 \leq i \leq n - 10, i \mod 2 = 0, \\
  i + 1, & 0 \leq i \leq n - 10, i \mod 2 = 1, \\
  2n - 16, & i = n - 9, \\
  2n - 13, & i = n - 8, \\
  2n - 12, & i = n - 7, \\
  2n - 15, & i = n - 6, \\
  2n - 5, & i = n - 5, \\
  2n - 9, & i = n - 4, \\
  2n - 4, & i = n - 3, \\
  2n - 1, & i = n - 2, \\
  2n - 10, & i = n - 1. 
\end{cases}$$
Lemma 2.22. $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-15} \cap \mathcal{N}_{-11} \cap \mathcal{N}_{-9}$.

Proof. In $\mathcal{N}_{-15} \cap \mathcal{N}_{-11} \cap \mathcal{N}_{-9}$, there are only two integers smaller than 52, namely 22, 28. Since 22 $\in \mathcal{N}_{-9} \cap \mathcal{N}_{-5} \cap \mathcal{N}_{-3}$, by Lemma 2.19, $P(22, 3)$ is prime. Since 28 $\in \mathcal{N}_{-19} \cap \mathcal{N}_{-15} \cap \mathcal{N}_{-13}$, by Lemma 2.8, $P(28, 3)$ is prime. Hence, we only consider even $n \geq 52$. And we define the function $f$ as follows:

Case 1. $n \equiv 0 \mod 7$. Let

$$f(v_i) = \begin{cases} 
  n + i - 10, & 0 \leq i \leq n - 12, \ i \mod 2 = 0, \\
  n - i, & 0 \leq i \leq n - 12, \ i \mod 2 = 1,
\end{cases}$$

$$f(u_i) = \begin{cases} 
  n + i - 11, & 0 \leq i \leq n - 12, \ i \mod 2 = 0, \\
  n + i, & 1 \leq i \leq n - 12, \ i \mod 2 = 1,
\end{cases}$$

Case 2. $n \equiv 3 \mod 7$. Let

$$f(v_i) = \begin{cases} 
  n - i - 10, & 0 \leq i \leq n - 12, \ i \mod 2 = 0, \\
  n - i, & 0 \leq i \leq n - 12, \ i \mod 2 = 1,
\end{cases}$$

$$f(u_i) = \begin{cases} 
  n - i - 11, & 0 \leq i \leq n - 12, \ i \mod 2 = 0, \\
  n - i, & 0 \leq i \leq n - 12, \ i \mod 2 = 1,
\end{cases}$$

Case 3. $n \equiv 5 \mod 7$. Let

$$f(v_i) = \begin{cases} 
  n - i - 10, & 0 \leq i \leq n - 12, \ i \mod 2 = 0, \\
  n - i, & 0 \leq i \leq n - 12, \ i \mod 2 = 1,
\end{cases}$$

$$f(u_i) = \begin{cases} 
  n - i - 11, & 0 \leq i \leq n - 12, \ i \mod 2 = 0, \\
  n - i, & 0 \leq i \leq n - 12, \ i \mod 2 = 1,
\end{cases}$$

Case 4. $n \not\equiv 0, 3, 5 \mod 7$, and $n \geq 52$. Let
Let $f(v_i)$ be defined as follows:

\[
\begin{align*}
\{ & n+i-10, \ 0 \leq i \leq n-12, i \mod 2 = 0, \\
& i, \ 0 \leq i \leq n-12, i \mod 2 = 1, \\
& 2n-1, \ i = n-11, \\
& 2n-2, \ i = n-10, \\
& 2n-9, \ i = n-9, \\
& 2n-10, \ i = n-8, \\
& 2n-5, \ i = n-7, \\
& 2n-4, \ i = n-6, \\
& 2n-7, \ i = n-5, \\
& 2n-14, \ i = n-4, \\
& 2n-21, \ i = n-3, \\
& 2n-18, \ i = n-2, \\
& 2n-19, \ i = n-1, \\
\}
\end{align*}
\]

In Figure 2.9(b), we show the prime labeling of $P(n,3)$, where $n = 8 \in N_{-1} \cap N_3 \cap N_5$.

**Lemma 2.24.** $P(n,3)$ is prime for even $n \in N_{-1} \cap N_3 \cap N_5$.

**Proof.** We define the function $f$ as follows:

\[
\begin{align*}
\{ & n-i-11, \ 0 \leq i \leq n-12, i \mod 2 = 0, \\
& i+1, \ 0 \leq i \leq n-12, i \mod 2 = 1, \\
& 2n, \ i = n-11, \\
& 2n-3, \ i = n-10, \\
& 2n-16, \ i = n-9, \\
& 2n-11, \ i = n-8, \\
& 2n-6, \ i = n-7, \\
& 2n-13, \ i = n-6, \\
& 2n-12, \ i = n-5, \\
& 2n-15, \ i = n-4, \\
& 2n-20, \ i = n-3, \\
& 2n-17, \ i = n-2, \\
& 2n-8, \ i = n-1. \\
\}
\end{align*}
\]
Let

\[
\begin{cases}
  i + 1, & 0 \leq i \leq n - 6, i \mod 2 = 0, \\
  n + i + 7, & 0 \leq i \leq n - 6, i \mod 2 = 1, \\
  n, & i = n - 5, \\
  n - 2, & i = n - 4, \\
  n + 5, & i = n - 3, \\
  n + 4, & i = n - 2,
\end{cases}
\]

\[
\begin{cases}
  f(v_i) = \\
  f(u_i) =
\end{cases}
\]

In Figure 2.10(a), we show the prime labeling of \( P(n, 3) \), where \( n = 12 \in \mathcal{N}_1 \cap \mathcal{N}_5 \cap \mathcal{N}_7 \).

![Figure 10: P(12, 3), 12 ∈ N1 ∩ N5 ∩ N7](image)

![Figure 10: P(10, 3), 10 ∈ N5 ∩ N7 ∩ N9](image)

**Lemma 2.25.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_3 \cap \mathcal{N}_7 \cap \mathcal{N}_9 \).

**Proof.**

In \( \mathcal{N}_3 \cap \mathcal{N}_7 \cap \mathcal{N}_9 \), there is only one integer smaller than 10, namely 4. Since \( 4 \in \mathcal{N}_1 \cap \mathcal{N}_3 \cap \mathcal{N}_7 \), by Lemma 2.16, \( P(4, 3) \) is prime. Hence, we only consider even \( n \geq 10 \). And we define the function \( f \) as follows:

\[
\begin{cases}
  i + 1, & 0 \leq i \leq n - 8, i \mod 2 = 0, \\
  n + i + 9, & 0 \leq i \leq n - 8, i \mod 2 = 1, \\
  n - 2, & i = n - 7, \\
  n, & i = n - 6, \\
  n - 3, & i = n - 5, \\
  n + 6, & i = n - 4, \\
  n + 7, & i = n - 3, \\
  n - 4, & i = n - 2,
\end{cases}
\]

\[
\begin{cases}
  f(v_i) = \\
  f(u_i) =
\end{cases}
\]

In Figure 2.10(b), we show the prime labeling of \( P(n, 3) \), where \( n = 10 \in \mathcal{N}_3 \cap \mathcal{N}_7 \cap \mathcal{N}_9 \).

**Lemma 2.26.** \( P(n, 3) \) is prime for even \( n \in \mathcal{N}_5 \cap \mathcal{N}_7 \cap \mathcal{N}_9 \).

**Proof.**

In \( \mathcal{N}_5 \cap \mathcal{N}_7 \cap \mathcal{N}_9 \), there is only one integer smaller than 32, namely 8. Since \( 8 \in \mathcal{N}_1 \cap \mathcal{N}_5 \cap \mathcal{N}_7 \), by Lemma 2.23, \( P(8, 3) \) is prime. Hence, we only consider even \( n \geq 32 \). And we define the function \( f \) as follows:
Case 1. $n \equiv 5 \mod 13$. Let

$$f(v_i) = \begin{cases} 
    i + 1, & 0 \leq i \leq n - 10, \mod 2 = 0, \\
    n + i + 11, & 0 \leq i \leq n - 10, \mod 2 = 1, \\
    n - 4, & i = n - 9, \\
    n - 7, & i = n - 8, \\
    n + 2, & i = n - 7, \\
    n + 1, & i = n - 6, \\
    n + 6, & i = n - 5, \\
    n + 5, & i = n - 4, \\
    n + 4, & i = n - 3, \\
    n - 1, & i = n - 2, \\
    n - 2, & i = n - 1,
\end{cases}$$

and

$$f(u_i) = \begin{cases} 
    i + 2, & 0 \leq i \leq n - 10, \mod 2 = 0, \\
    n + i + 10, & 0 \leq i \leq n - 10, \mod 2 = 1, \\
    n - 5, & i = n - 9, \\
    n - 6, & i = n - 8, \\
    n + 9, & i = n - 7, \\
    n, & i = n - 6, \\
    n + 7, & i = n - 5, \\
    n + 10, & i = n - 4, \\
    n + 3, & i = n - 3, \\
    n + 8, & i = n - 2, \\
    n - 3, & i = n - 1.
\end{cases}$$

Case 2. $n \not\equiv 5 \mod 13$. Let

$$f(v_i) = \begin{cases} 
    i + 1, & 0 \leq i \leq n - 10, \mod 2 = 0, \\
    n + i + 11, & 0 \leq i \leq n - 10, \mod 2 = 1, \\
    n - 4, & i = n - 9, \\
    n + 1, & i = n - 8, \\
    n + 2, & i = n - 7, \\
    n - 1, & i = n - 6, \\
    n - 2, & i = n - 5, \\
    n - 3, & i = n - 4, \\
    n + 6, & i = n - 3, \\
    n + 9, & i = n - 2, \\
    n + 4, & i = n - 1,
\end{cases}$$

and

$$f(u_i) = \begin{cases} 
    i + 2, & 0 \leq i \leq n - 10, \mod 2 = 0, \\
    n + i + 10, & 0 \leq i \leq n - 10, \mod 2 = 1, \\
    n - 5, & i = n - 9, \\
    n - 6, & i = n - 8, \\
    n + 9, & i = n - 7, \\
    n, & i = n - 6, \\
    n + 7, & i = n - 5, \\
    n + 10, & i = n - 4, \\
    n + 3, & i = n - 3, \\
    n + 8, & i = n - 2, \\
    n + 5, & i = n - 1.
\end{cases}$$

In Figure 2.11(a), we show the prime labeling of $P(n, 3)$, where $n = 32 \in \mathcal{N}_5 \cap \mathcal{N}_9 \cap \mathcal{N}_{11}$.

![Figure 11](image_url)

**Lemma 2.27.** $P(n, 3)$ is prime for even $n \in \mathcal{N}_7 \cap \mathcal{N}_{11} \cap \mathcal{N}_{13}$.

**Proof.**

In $\mathcal{N}_7 \cap \mathcal{N}_{11} \cap \mathcal{N}_{13}$, there is only one integer smaller than 30, namely 6. Since $6 \in \mathcal{N}_7 \cap \mathcal{N}_9 \cap \mathcal{N}_7$, by Lemma 2.24, $P(6, 3)$ is prime. Hence, we only consider even $n \geq 30$. And we define the function $f$ as follows:
Let

\[
\begin{align*}
  f(v_i) &= \begin{cases} 
    i + 1, & 0 \leq i \leq n - 12, i \mod 2 = 0, \\
    n + i + 13, & 0 \leq i \leq n - 12, i \mod 2 = 1, \\
    n - 8, & i = n - 11, \\
    n - 9, & i = n - 10, \\
    n - 4, & i = n - 9, \\
    n - 5, & i = n - 8, \\
    n - 2, & i = n - 7, \\
    n + 1, & i = n - 6, \\
    n, & i = n - 5, \\
    n - 1, & i = n - 4, \\
    n + 6, & i = n - 3, \\
    n + 11, & i = n - 2, \\
    n + 10, & i = n - 1, 
  \end{cases} \\
  f(u_i) &= \begin{cases} 
    i + 2, & 0 \leq i \leq n - 12, i \mod 2 = 0, \\
    n + i + 12, & 0 \leq i \leq n - 12, i \mod 2 = 1, \\
    n - 7, & i = n - 11, \\
    n + 2, & i = n - 10, \\
    n - 3, & i = n - 9, \\
    n - 6, & i = n - 8, \\
    n + 3, & i = n - 7, \\
    n + 4, & i = n - 6, \\
    n + 7, & i = n - 5, \\
    n + 8, & i = n - 4, \\
    n + 5, & i = n - 3, \\
    n + 12, & i = n - 2, \\
    n + 9, & i = n - 1, 
  \end{cases}
\end{align*}
\]

In Figure 2.11(b), we show the prime labeling of \( P(n, 3) \), where \( n = 30 \in \mathcal{N}_7 \cap \mathcal{N}_9 \cap \mathcal{N}_{13} \).

From the Lemmas 2.4 - 2.27, Theorem 2.2 holds. Furthermore, we have the following conjecture

**Conjecture 2.28.** \( P(n, 3) \) is prime for all even \( n \).

Since \( n \in \mathcal{N} \cup \mathcal{N}^* \) for any even \( n \leq 100 \), by Theorem 2.2 and Table 2.1, we have Conjecture 2.28 holds for even \( n \leq 100 \).

**References**


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