(\(k, r\))-Domination in Graphs

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Abstract

Let \(G = (V, E)\) be a simple graph. A subset \(D\) of \(V(G)\) is a \((k, r)\)-dominating set if for every vertex \(v \in V - D\), there exists at least \(k\) vertices in \(D\) which are at a distance utmost \(r\) from \(v\) in \([1]\). The minimum cardinality of a \((k, r)\)-dominating set of \(G\) is called the \((k, r)\)-domination number of \(G\) and is denoted by \(\gamma_{(k, r)}(G)\). In this paper, minimal \((k, r)\)-dominating sets are characterized. It is proved that Vizing conjecture does not hold in the case of \((k, r)\)-domination.

Mathematics Subject Classification: 05C69

Keywords: \((k, r)\)-domination number

1 Introduction

Consider a network in which there are signal transmitting centres and signal receiving centres. The receiving centres may hope to get good signals if the transmitting centres are at a distance of at most \(r\) (say) from the receiving centres. In the event of failures of signal transmitting centres, to retain the integrity of the network one can impose an additional condition that, for each non-transmitting centre there are at least \(k\)-transmitting centres, which send signals to the non-transmitting centre. \(k\) may be sufficiently large positive integer to allow for adequate security of transmission in all likely events of a break down in reliable communications. To find a graph model for this, Michael A. Henning et al, [2] introduced the concept of \((k, r)\)-domination.
We consider only finite simple graphs. In the first section, we start with the definition by Henning et al, introduce \((k, r)\)-neighbourhood of a vertex and find the \((k, r)\)-domination number of a standard graph. The second section deals with the minimal \((k, r)\) -dominating sets. Also, a chain connecting \(\gamma_{(1,r)}(G)\) with \(\gamma_{(k,1)}(G)\) is found out. For an even path of length \(2t\), the relation between \(\gamma_{2,t-2}, \gamma_{2,t-1}, \ldots, \gamma_{2,2t}\) is determined. The third section deals with Vizing conjecture in the case of \((k, r)\)-domination. Conclusion is given at the end.

2 \((k, r)\)-domination:

**Definition 2.1** Let \(G = (V,E)\) be a graph. Let \(r, k \geq 1\) be integers. A subset \(D\) of \(V\) is a \((k, r)\)-dominating set if for every vertex \(u\) in \(V - D\), there exists at least \(k\) vertices in \(D\) which are at a distance at most \(r\) from \(u\). The minimum (maximum) cardinality of a minimal \((k, r)\)-dominating set is called a \((k, r)\)-domination number of \(G\) (upper \((k, r)\)-domination number of \(G\)) and is denoted by \(\gamma_{(k,r)}(G)\).

**Definition 2.2** The open \(r\)-neighbourhood \(N_r(v)\) of a vertex \(v\) in a graph \(G\) is defined by \(N_r(v) = \{u \in V(G) : 0 < d(u,v) \leq r\}\) and its closed \(r\)-neighbourhood is \(N_r[v] = N_r(v) \cup \{v\}\). The \(r\)-degree of \(v\) in \(G\), \(\deg_r(v)\) is given by \(|N_r(v)|\), while \(\Delta_r(G)\) and \(\delta_r(G)\) denote the maximum and minimum \(r\)-degree among all the vertices of \(G\) respectively.

**Definition 2.3** Given the positive integers \(k\) and \(r\), the \((k, r)\)-neighbourhood of a vertex \(u \in V(G)\), denoted by \(N_{(k,r)}(u)\) and is defined as

\[
N_{(k,r)}(u) = \begin{cases} 
N_r(u), & \text{if } |N_r(u)| \geq k \\
\emptyset, & \text{otherwise} 
\end{cases}
\]

The closed \((k, r)\)-neighbourhood is 

\[
N_{(k,r)}[u] = N_{(k,r)}(u) \cup \{u\}.
\]

A vertex \(u \in V\) is a \((k, r)\)-isolate if \(N_{(k,r)}(u) = \emptyset\).

**Definition 2.4** Given the positive integers \(k\) and \(r\) and a subset \(D\) of \(V\), the \((k, r, D)\)-neighbourhood of a vertex \(u \in V(G)\), denoted by \(N_{(k,r,D)}(u)\) is defined as

\[
N_{(k,r,D)}(u) = \begin{cases} 
N_r(u) \cap D, & \text{if } |N_r(u) \cap D| \geq k \\
\emptyset, & \text{otherwise} 
\end{cases}
\]

The closed \((k, r, D)\)-neighbourhood is 

\[
N_{(k,r,D)}[u] = N_{(k,r,D)}(u) \cup \{u\}.
\]

A vertex \(u \in V\) is a \((k, r, D)\)-isolate if \(N_{(k,r,D)}(u) = \emptyset\).
When \( r = 1 \) and \( k = 1 \), \( D = \{ v_4, v_5 \} \). Therefore \( \gamma_{(1,1)}(G) = 2 \). When \( r = 1 \) and \( k = 2 \), \( D = \{ v_1, v_2, v_3, v_6, v_7, v_8 \} \). Hence \( \gamma_{(2,1)}(G) = 6 \). When \( r = 1 \) and \( k = 3 \), \( D = \{ v_1, v_2, v_3, v_6, v_7, v_8 \} \). Therefore \( \gamma_{(3,1)}(G) = 6 \). It can be shown that \( \gamma_{(4,1)}(G) = 7 \), and \( \gamma_{(k,1)}(G) = 8 \), for every \( k \geq 5 \). When \( r = 1 \) and \( k = 2 \), \( D = \{ v_4 \} \). Therefore \( \gamma_{(1,2)}(G) = 1 \). When \( r = 2 \) and \( k = 2 \), \( D = \{ v_4, v_5 \} \). Therefore \( \gamma_{(2,2)}(G) = 2 \). When \( r = 2 \) and \( k = 3 \), \( D = \{ v_1, v_2, v_3, v_6, v_7, v_8 \} \). Therefore \( \gamma_{(3,2)}(G) = 4 \).

When \( r = 2 \) and \( k = 4, 5 \) and 6, \( D = \{ v_1, v_2, v_3, v_6, v_7, v_8 \} \). Therefore \( \gamma_{(k,2)}(G) = 6 \). Further, \( \gamma_{(k,2)}(G) = k \), if \( k = 7 \) and \( k = 8 \). \( \gamma_{(k,r)}(G) = k \), if \( r \geq 3 \).

**Remark 2.5** Let \( G = (V,E) \) be a connected graph. Then \( V \) itself is a \((k,r)\)-dominating set. Therefore the existence of \((k,r)\)-dominating set is guaranteed for any graph \( G \).

**Theorem 2.6** Let \( G \) be a graph. Then \( k \leq \gamma_{(k,r)}(G) \leq n \) and these bounds are sharp.

**Proof:** Let \( G \) be a graph. For a vertex to be \((k,r)\)-dominated, there must be at least \( k \)-vertices in any \((k,r)\)-dominating set. Therefore \( k \leq \gamma_{(k,r)}(G) \). It is obvious that \( V \) forms a \((k,r)\)-dominating set and therefore any \((k,r)\)-dominating set contains at most \( n \)vertices. Therefore \( \gamma_{(k,r)}(G) \leq n \). The lower bound is sharp if \( r = \text{diam}(G) \) and the upper bound is sharp if \( k > \Delta_r(G) \).

**Theorem 2.7** If \( r = \text{diam}(G) \), then \( \gamma_{(k,r)}(G) = k \).

**Proof:** If \( r = \text{diam}(G) \), then every vertex of \( G \) is at a distance \( \leq r \) with every other vertex of \( G \). Any \( k \)-element subset of \( V(G) \) is a \((k,r)\)-dominating set. But any \((k,r)\)-dominating set has at least \( k \)-elements. Therefore \( \gamma_{(k,r)}(G) = k \).

**Remark 2.8** The converse of the above theorem is not be true. \( \gamma_{(k,r)}(G) = k \) does not imply that \( r = \text{diam}(G) \).

\[
G: \\
\begin{array}{c|c|c}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]

It can be easily verified that \( \gamma_{(3,2)}(G) = 3 = k \). But, \( \text{diam}(G) = 3 > r = 2 \).

**Theorem 2.9** \( k > \Delta_r(G) \) if and only if \( \gamma_{(k,r)}(G) = n \).
Proof: Suppose $k > \Delta_r(G)$. Let $D$ be a $\gamma_{(k,r)}$-set of $G$.

Claim: $\gamma_{(k,r)}(G) = n$. (ie) $V - D = \emptyset$.

If not, let $x \in V - D$. Then there exist at least $k$-vertices $u_1, u_2, \ldots u_l \in D$, where $l \geq k$ and $d(u_i, x) \leq r$, for all $i = 1$ to $l, l \geq k$. Therefore $k \leq l \leq \Delta_r(G)$, a contradiction. Therefore $\gamma_{(k,r)}(G) = n$. Conversely, let $D$ be a $\gamma_{(k,r)}$-set of $G$ and $|D| = \gamma_{(k,r)}(G) = n$. Claim: $k > \Delta_r(G)$.

On the contrary, suppose that $k \leq \Delta_r(G)$. Let $x$ be a vertex of maximum $r$-degree in $G$ and let $N_r(x) = \{u_1, u_2, \ldots u_{\Delta_r(G)}\}$. Then $x$ has at least $kr$-neighbours. Therefore $V - \{x\}$ is a $(k, r)$-dominating set. Therefore $\gamma_{(k,r)}(G) \leq n - 1$, a contradiction. Hence $k > \Delta_r(G)$.

$(k, r)$-domination number for Standard Graphs:

1. $\gamma_{(k,r)}(K_n) = k$ for all $k$ and $r$

2. $\gamma_{(k,r)}(K_{(1,n)}) = \begin{cases} 1 & \text{if } k = 1 \text{ and } r = 1 \\ n & \text{if } r = 1 \text{ and } 2 \leq k \leq n \\ k & \text{if } r \geq 2 \text{ and } \text{for all } k. \end{cases}$

3. $\gamma_{(k,r)}(K_{(m,n)}) = \begin{cases} \min\{2k, z\}, & \text{if } r = 1 \text{ and } k \leq z \\ z', & \text{if } r = 1 \text{ and } z < k \leq z' \\ m + n, & \text{if } r = 1 \text{ and } k > z' \\ k, & \text{if } r \geq 2 \text{ and } 1 \leq k \leq m + n \end{cases}$ where $z = \min\{m, n\}$ and $z' = \max\{m, n\}$

4. $\gamma_{(k,r)}(W_n) = \begin{cases} 1 & \text{if } r = 1 \text{ and } k = 1 \\ \lfloor (n-1)/2 \rfloor & \text{if } r = 1 \text{ and } k = 2 \\ \lfloor (n-1)/2 \rfloor + 1 & \text{if } r = 1 \text{ and } k = 3 \\ n - 1 & \text{if } r = 1, \ k \geq 4 \\ k, & \text{if } r \geq 2, \ 1 \leq k \leq n \end{cases}$

5. $\gamma_{(k,r)}(C_n) = \begin{cases} \lfloor n/3 \rfloor & \text{if } r = 1 \text{ and } k = 1 \\ \lfloor n/2 \rfloor, & \text{if } r = 1 \text{ and } k = 2 \\ n & \text{if } r = 1, \ k \geq 3 \\ \lfloor n/(2r+1) \rfloor & \text{if } r \geq 2 \text{ and } k = 1 \\ \lfloor n/(k+r-1) \rfloor & \text{if } r \geq 2, \text{ and } k = 2 \end{cases}$

Remark 2.10 If $D$ is a $(k, r)$-dominating set, then any superset of $D$ is also a $(k, r)$-dominating set. That is, $(k, r)$-domination has the superhereditary property.
Proposition 2.11 For any graph $G$, $D$ is a $(k, r)$-dominating set of $G$ if and only if $\bigcup_{T} \left( \bigcap_{u_i \in T} N_r(u_i) \right) \cup D = V$, where $T$ is a $k$-subset of $D$.

Proof: Let $D$ be a $(k, r)$-dominating set. It is clear that $\bigcup_{T} \left( \bigcap_{u_i \in T} N_r(u_i) \right) \cup D \subseteq V$. Let $u \in V$. If $u \in D$, then there is nothing to prove. If $u \notin D$, then there exists at least $k$ elements $u_1, u_2, \ldots, u_l$ in $D$, where $l \geq k$ such that $d(u, u_i) \leq r$. Then $u \in N_r(u_i)$ for all $i, 1 \leq i \leq l$ which implies that $u \in \bigcup_{T} \left( \bigcap_{u_i \in T} N_r(u_i) \right)$, where $T$ is a $k$-subset of $D$. Conversely, let $\bigcup_{T} \left( \bigcap_{u_i \in T} N_r(u_i) \right) \cup D = V$. Then we will prove that $D$ is a $(k, r)$-dominating set. Let $u \in V - D$. Then $u \in \bigcup_{T} \left( \bigcap_{u_i \in T} N_r(u_i) \right)$ which implies that $u \in \left( \bigcap_{u_i \in T} N_r(u_i) \right)$, for some $k$-subset $T$ of $D$ and hence $D$ is a $(k, r)$-dominating set.

3 Minimal $(k, r)$-dominating sets

Definition 3.1 A $(k, r)$-dominating set $D$ of a graph $G$ is said to be minimal if no proper subset of $D$ is a $(k, r)$-dominating set of $G$.

Proposition 3.2 A $(k, r)$-dominating set $D$ is a minimal $(k, r)$-dominating set if and only if for each vertex $u \in D$, one of the following two conditions hold. a) $u$ is a $(k, r, D)$-isolate. b) There exists a vertex $v \in V - D$ for which $|N_r(v) \cap D| = k$ and $u \in N_r(v) \cap D$.

Proof: Let $D$ be a minimal $(k, r)$-dominating set. Suppose there exists a vertex $u \in D$ which is not a $(k, r, D)$-isolate and for every $v \in V - D$, either $|N_r(v) \cap D| > k$ or $u \notin N_r(v) \cap D$. Consider $D' = D - \{u\}$. Since $u$ is at a distance $\leq r$ with at least $k$ vertices of $D'$, $D'$ is a $(k, r)$-dominating set, which is a contradiction to the minimality of $D$. Hence for each vertex $u \in D$, one of the two conditions hold.

Conversely, let $D$ be a $(k, r)$-dominating set satisfying (a) and (b). Consider $D' = D - \{u\}$ for an arbitrary vertex $u \in D$. If (a) holds, then $|N_r(u) \cap D'| < k$, which implies that $D'$ is not a $(k, r)$-dominating set. If (b) holds, then the set $D'$ would not $(k, r)$-dominate $u$. Hence $D$ is a minimal $(k, r)$-dominating set.

Remark 3.3 If $G$ has no $(k, r)$-isolates and if $D$ is a minimal $(k, r)$-dominating set, then $V - D$ need not be a $(k, r)$-dominating set.
Proof: The proof is by the following construction.

\[ \gamma(2, t-2)(P_n) > \gamma(2, t-1)(P_n) > \gamma(2, t)(P_n) = \gamma(2, t+1)(P_n) = \cdots = \gamma(2, 2t)(P_n). \]

Proof: Let \( n = 2t \). Let \( V(P_n) = \{v_1, v_2, v_3, \ldots, v_{2t}\} \). Clearly \( \{v_{t-2}, v_{t-1}, v_{t+2}, v_{t+3}\} \) is a \((2, t-2)\)-dominating set of \( P_n \). Let \( \{v_i, v_j, v_k\} \) be a \((2, t-2)\)-dominating set, \( 1 \leq i < j < k \leq 2t \). Then any two of \( d(v_1, v_i) \), \( d(v_1, v_j) \) and \( d(v_1, v_k) \) are less than or equal to \( t-2 \). That is any two of \( i-1, j-1, k-1 \) are less than or equal to \( t-2 \). Therefore, any two of \( i, j, k \) are less than or equal to \( t-1 \). Let \( i \leq t-1 \) and \( j \leq t-1 \). Maximum value of \( j \) is \( t-1 \). Then \( d(v_{2t}, v_j) = 2t-j \geq 2t-(t-1) = t+1 \). Similarly, \( d(v_{2t}, v_i) \geq t+1 \). Therefore \( v_{2t} \) is not \((2, t-2)\)-dominated by \( v_i \) and \( v_j \), a contradiction. In a similar manner, we can prove that \( \{v_i, v_j, v_k\} \) is not a \((2, t-2)\)-dominating set in other cases also. Therefore \( \gamma(2, t-2)(P_n) = 4 \). Hence the remark.

Proposition 3.8 Given positive integers \( k \) and \( r \), there exists a connected graph \( G \) with \( \gamma(k, r)(G) = k \) and \( \text{diam}(G) = r + 1 \).

Proof: The proof is by the following construction.
Let $D = \{v_1, v_2, \ldots, v_k\}$. Let $v_1, w_1, w_2, \ldots, w_r = u_1$ be a shortest path between $v_1$ and $u_1$. Let $v_2, x_1, x_2, \ldots, x_r = u_2$ be a shortest path between $v_2$ and $u_2$. Let $u_1, u_2$ be adjacent to $v_3, v_4, \ldots, v_k$. Let $u_1$ be adjacent to $v_2$ and $u_2$ be adjacent to $v_1$. Now $diam(G) = r + 1$. $D$ is a $(k, r)$-dominating set of $G$ and $|D| = k$. Therefore $\gamma(k, r)(G) \leq |D| = k$. But $k \leq \gamma(k, r)(G)$. Therefore $D$ is a $\gamma(k, r)$-set of $G$ and $r < diam(G) = r + 1$.

4 Vizing Conjecture:

For any graph $G$ and $H$, $\gamma(G \times H) \geq \gamma(G)\gamma(H)$. But in the case of $(k, r)$-domination,

$\gamma(k, r)(G \times H) < \gamma(k, r)(G)\gamma(k, r)(H)$, for some $k$ and $r$.

In the above example, $\{1, 3, 5, 7, 9, 11, 13, 15\}$ is a $\gamma(3,1)$ set of $G$. $\gamma(3,1)(P_3 \times P_5) = 8$. $\gamma(3,1)(P_3) = 3$. $\gamma(3,1)(P_5) = 5$. Hence, $\gamma(3,1)(P_3 \times P_5) < \gamma(3,1)(P_3)\gamma(3,1)(P_5)$. In the above example, $\{6, 7, 9, 10\}$ is a $\gamma(2,2)$ set of $G$. $\gamma(2,2)(P_3 \times P_5) = 4$. $\gamma(2,2)(P_3) = 2$. $\gamma(2,2)(P_5) = 3$. Hence $\gamma(2,2)(P_3 \times P_5) < \gamma(2,2)(P_3)\gamma(2,2)(P_5)$.

In the above example, $\{3, 8, 13\}$ is a $\gamma(2,3)$ set of $G$. $\gamma(2,3)(P_3 \times P_5) = 3$. $\gamma(2,3)(P_3) = 2$. $\gamma(2,3)(P_5) = 2$. Hence $\gamma(2,3)(P_3 \times P_5) < \gamma(2,3)(P_3)\gamma(2,3)(P_5)$.

5 Conclusion:

We have made a study of $(k, r)$-domination. It is further continued in our subsequent investigations in this direction. Facility location problems as con-
sidered in [5] make use of \((k,r)\)-domination. Other applications are also attempted.

References


Received: February, 2011