Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 29, 1439 - 1446

(k, r)-Domination in Graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A subset $D$of $V(G)$ is a (k, r)-dominating set if for every vertex$v \in V - D$, there exists at least k vertices in D which are at a distance utmost r from v in **[1]**. The minimum cardinality of a (k, r)-dominating set of G is called the (k, r)-domination number of G and is denoted by $\gamma_{(k,r)}(G)$. In this paper, minimal (k, r)-dominating sets are characterized. It is proved that Vizing conjecture does not hold in the case of (k, r)-domination.

Mathematics Subject Classification: 05C69

Keywords: (k, r)-domination number

1 Introduction

Consider a network in which there are signal transmitting centres and signal receiving centres. The receiving centres may hope to get good signals if the transmitting centres are at a distance of at most r (say) from the receiving centres. In the event of failures of signal transmitting centres, to retain the integrity of the network one can impose an additional condition that, for each non-transmitting centre there are at least k-transmitting centres, which send signals to the non-transmitting centre. k may be sufficiently large positive integer to allow for adequate security of transmission in all likely events of a break down in reliable communications. To find a graph model for this, Michael A. Henning et al, **[2]** introduced the concept of (k, r)-domination.

We consider only finite simple graphs.In the first section, we start with the definition by Henning et al, introduce (k, r)-neighbourhood of a vertex and find the (k, r)-domination number of standard graphs. The second section deals with the minimal (k, r) -dominating sets. Also, a chain connecting $\gamma_{(1,r)}(G)$ with $\gamma_{(k,1)}(G)$ is found out. For an even path of length $2t$, the relation between $\gamma_{2,t-2}, \gamma_{2,t-1}, \ldots, \gamma_{2,2t}$ is determined. The third section deals with Vizing conjecture in the case of (k, r)-domination. Conclusion is given at the end.

2 (k, r)-domination:

Definition 2.1 *Let $G = (V, E)$ be a graph. Let $r, k \geq 1$ be integers. A subset D of V is a (k, r)-dominating set if for every vertex u in $V - D$, there exists at least k vertices in D which are at a distance at most r from u. The* **minimum (maximum)** *cardinality of a minimal (k, r)-dominating set is called a* **(k, r)-domination number of G (upper (k, r)-domination number of G)** *and is denoted by $\gamma_{(k,r)}(G)(\Gamma_{(k,r)}(G))$.*

Definition 2.2 *The **open r-neighbourhood** $N_r(v)$ of a vertex v in a graph G is defined by $N_r(v) = \{u \in V(G) : 0 < d(u, v) \leq r\}$ and its **closed r-neighbourhood** is $N_r[v] = N_r(v) \cup \{v\}$. The r-degree of v in G, $deg_r(v)$ is given by $|N_r(v)|$, while $\Delta_r(G)$ and $\delta_r(G)$ denote the maximum and minimum r-degree among all the vertices of G respectively.*

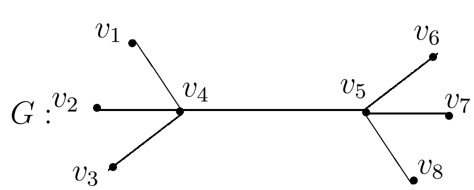
Definition 2.3 *Given the positive integers k and r, the* **(k, r)-neighbourhood of a vertex** *$u \in V(G)$, denoted by $N_{(k,r)}(u)$ and is defined as*

$$N_{(k,r)}(u) = \begin{cases} N_r(u), & if |N_r(u)| \geq k \\ \emptyset, & otherwise \end{cases}.$$

The **closed (k, r)-neighbourhood** *is $N_{(k,r)}[u] = N_{(k,r)}(u) \cup \{u\}$. A vertex $u \in V$ is a* **(k, r)-isolate** *if $N_{(k,r)}(u) = \emptyset$.*

Definition 2.4 *Given the positive integers k and r and a subset D of V, the* **(k, r, D)-neighbourhood of a vertex** *$u \in V(G)$, denoted by $N_{(k,r,D)}(u)$ is defined as* $N_{(k,r,D)}(u) = \begin{cases} N_r(u) \cap D, & if\ |N_r(u) \cap D| \geq k \\ \emptyset, & otherwise \end{cases}$

The **closed (k, r, D)-neighbourhood** *is $N_{(k,r,D)}[u] = N_{(k,r,D)}(u) \cup \{u\}$. A vertex $u \in V$ is a* **(k, r, D)-isolate** *if $N_{(k,r,D)}(u) = \emptyset$.*



When $r = 1$ and $k = 1$, $D = \{v_4, v_5\}$. Therefore $\gamma_{(1,1)}(G) = 2$. When $r = 1$ and $k = 2$, $D = \{v_1, v_2, v_3, v_6, v_7, v_8\}$. Hence $\gamma_{(2,1)}(G) = 6$. When $r = 1$ and $k = 3$, $D = \{v_1, v_2, v_3, v_6, v_7, v_8\}$. Therefore $\gamma_{(3,1)}(G) = 6$. It can be shown that $\gamma_{(4,1)}(G) = 7$, and $\gamma_{(k,1)}(G) = 8$, for every $k \geq 5$. When $r = 2$ and $k = 1$, $D = \{v_4\}$. Therefore $\gamma_{(1,2)}(G) = 1$. When $r = 2$ and $k = 2$, $D = \{v_4, v_5\}$. Therefore $\gamma_{(2,2)}(G) = 2$. When $r = 2$ and $k = 3$, $D = \{v_4, v_5, v_1, v_6\}$. Therefore $\gamma_{(3,2)}(G) = 4$.
When $r = 2$ and $k = 4, 5$ and 6, $D = \{v_1, v_2, v_3, v_6, v_7, v_8\}$. Therefore $\gamma_{(k,2)}(G) = 6$. Further, $\gamma_{(k,2)}(G) = k,\ if\ k = 7\ and\ k = 8$. $\gamma_{(k,r)}(G) = k$, if $r \geq 3$.

Remark 2.5 *Let $G = (V, E)$ be a connected graph. Then V itself is a (k, r)-dominating set. Therefore the existence of (k, r)-dominating set is guarenteed for any graph G.*

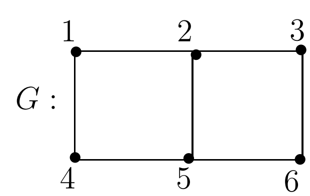
Theorem 2.6 *Let G be a graph. Then $k \leq \gamma_{(k,r)}(G) \leq n$ and these bounds are sharp.*

Proof: Let G be a graph. For a vertex to be (k, r)-dominated, there must be at least k-vertices in any (k, r)-dominating set. Therefore $k \leq \gamma_{(k,r)}(G)$. It is obvious that V forms a (k, r)-dominating set and therefore any (k, r)-dominating set contains at most $n$vertices. Therefore $\gamma_{(k,r)}(G) \ \leq n$. The lower bound is sharp if $r = diam(G)$ and the upper bound is sharp if $k > \Delta_r(G)$.

Theorem 2.7 *If $r = diam(G)$, then $\gamma_{(k,r)}(G) = k$.*

Proof: If $r = diam(G)$, then every vertex of G is at a distance $\leq r$ with every other vertex of G. Any k-element subset of $V(G)$ is a (k, r)-dominating set. But any (k, r)-dominating set has at least k-elements. Therefore $\gamma_{(k,r)}(G) = k$.

Remark 2.8 *The converse of the above theorem is not be true. $\gamma_{(k,r)}(G) = k$ does not imply that $r = diam(G)$.*



It can be easily verified that $\gamma_{(3,2)}(G) = 3 = k$. But, $diam(G) = 3 > r = 2$.

Theorem 2.9 *$k > \Delta_r(G)$ if and only if $\gamma_{(k,r)}(G) = n$.*

Proof: Suppose $k > \Delta_r(G)$. Let D be a $\gamma_{(k,r)}$-set of G.
Claim: $\gamma_{(k,r)}(G) = n$. (ie) $V - D = \emptyset$.

If not, let $x \in V - D$. Then there exist at least k-vertices $u_1, u_2, \ldots u_l \in D$, where $l \geq k$ and $d(u_i, x) \leq r$, for all $i = 1$ to $l, l \geq k$. Therefore $k \leq l \leq \Delta_r(G)$, a contradiction. Therefore $\gamma_{(k,r)}(G) = n$. Conversely, let D be a $\gamma_{(k,r)}$-set of G and $|D| = \gamma_{(k,r)}(G) = n$. *Claim:* $k > \Delta_r(G)$.

On the contrary, suppose that $k \leq \Delta_r(G)$. Let x be a vertex of maximum r-degree in G and let $N_r(x) = \{u_1, u_2, \ldots u_{(\Delta_r(G))}\}$. Then x has at least k r-neighbours. Therefore $V - \{x\}$ is a (k,r)-dominating set. Therefore $\gamma_{(k,r)}(G) \leq n - 1$, a contradiction. Hence $k > \Delta_r(G)$.

(k,r)-domination number for Standard Graphs:

1. $\gamma_{(k,r)}(K_n) = k$ for all k and r

2. $$\gamma_{(k,r)}(K_{(1,n)}) = \begin{cases} 1 \ if \ k = 1 \ and \ r = 1 \\ n \ if \ r = 1 \ and \ 2 \leq k \leq n \\ k \ \ if \ r \geq 2 \ and \ for \ all \ k. \end{cases}$$

3. $$\gamma_{(k,r)}(K_{(m,n)}) = \begin{cases} \min\{2k, z\}, & \text{if } r = 1 \text{ and } k \leq z \\ z', & \text{if } r = 1 \text{ and } z < k \leq z' \\ m + n, & \text{if } r = 1 \text{ and } k > z' \\ k, & \text{if } r \geq 2 \text{ and } 1 \leq k \leq m + n \end{cases}$$ where $z = \min\{m, n\}$ and $z' = \max\{m, n\}$

4. $$\gamma_{(k,r)}(W_n) = \begin{cases} 1 \ \ if \ r = 1 \ and \ k = 1 \\ \lceil (n-1)/2 \rceil \ \ if \ r = 1 \ and \ k = 2 \\ \lceil (n-1)/2 \rceil + 1 \ \ if \ r = 1 \ and \ k = 3 \\ n - 1 \ \ if \ r = 1, \ k \geq 4 \\ k, \ if \ r \geq 2, \ 1 \leq k \leq n \end{cases}$$

5. $$\gamma_{(k,r)}(C_n) = \begin{cases} \lceil n/3 \rceil \ \ if \ r = 1 \ and \ k = 1 \\ \lceil n/2 \rceil, \ if \ r = 1 \ and \ k = 2 \\ n \ \ if \ r = 1, \ k \geq 3 \\ \lceil n/(2r+1) \rceil \ \ if \ r \geq 2 \ and \ k = 1 \\ \lceil n/(k+r-1) \rceil \ \ if \ r \geq 2, \ and \ k = 2 \end{cases}$$

Remark 2.10 *If D is a (k,r)-dominating set, then any superset of D is also a (k,r)-dominating set. That is, (k,r)-domination has the superhereditary property.*

Proposition 2.11 *For any graph G, D is a (k, r)-dominating set of G if and only if* $\underset{T}{\cup}\left(\underset{u_i \in T}{\cap} N_r(u_i)\right) \cup D = V$, *where T is a k-subset of D.*

Proof: Let D be a (k, r)-dominating set. It is clear that $\underset{T}{\cup}\left(\underset{u_i \in T}{\cap} N_r(u_i)\right) \cup D \subseteq V$. Let $u \in V$. If $u \in D$, then there is nothing to prove. If $u \notin D$, then there exists at least k elements $u_1, u_2, \ldots, u_l$ in D, where $l \geq k$ such that $d(u_i, u) \leq r$. Then $u \in N_r(u_i)$ for all $i, 1 \leq i \leq l$ which implies that $u \in \underset{T}{\cup}\left(\underset{u_i \in T}{\cap} N_r(u_i)\right)$, where T is a k-subset of D. Conversely, let $\underset{T}{\cup}\left(\underset{u_i \in T}{\cap} N_r(u_i)\right) \cup D = V$. Then we will prove that D is a (k, r)-dominating set. Let $u \in V - D$. Then $u \in \underset{T}{\cup}\left(\underset{u_i \in T}{\cap} N_r(u_i)\right)$ which implies that $u \in (\underset{u_i \in T}{\cap} N_r(u_i))$, for some k-subset T of D and hence D is a (k, r)-dominating set.

3 Minimal (k, r)-dominating sets

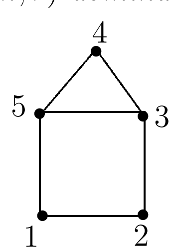
Definition 3.1 *A (k, r)-dominating set D of a graph G is said to be minimal if no proper subset of D is a (k, r)-dominating set of G.*

Proposition 3.2 *A (k, r)-dominating set D is a minimal (k, r)-dominating set if and only if for each vertex $u \in D$, one of the following two conditions hold. a) u is a (k, r, D)-isolate. b) There exists a vertex $v \in V - D$ for which $|N_r(v) \cap D| = k$ and $u \in N_r(v) \cap D$.*

Proof: Let D be a minimal (k, r)-dominating set. Suppose there exists a vertex $u \in D$ which is not a (k, r, D)-isolate and for every $v \in V - D$, either $|N_r(v) \cap D| > k$ or $u \notin N_r(v) \cap D$. Consider $D' = D - \{u\}$. Since u is at a distance $\leq r$ with at least k vertices of D', D' is a (k, r)-dominating set, which is a contradiction to the minimality of D. Hence for each vertex $u \in D$, one of the two conditions hold.

Conversely, let D be a (k, r)-dominating set satisfying (a) and (b). Consider $D' = D - \{u\}$ for an arbitrary vertex $u \in D$. If (a) holds, then $|N_r(u) \cap D'| < k$, which implies that D' is not a (k, r)-dominating set. If (b) holds, then the set D' would not (k, r)-dominate u. Hence D is a minimal (k, r)-dominating set.

Remark 3.3 *If G has no (k, r)-isolates and if D is a minimal (k, r)-dominating set, then $V - D$ need not be a (k, r)-dominating set.*



$N_1(1) = \{2, 5\}$; $N_1(2) = \{1, 3\}$; $N_1(3) = \{2, 5, 4\}$. $N_1(4) = \{3, 5\}$; $N_1(5) = \{1, 4, 3\}$. G has no $(2, 1)$-isolates and $D = \{2, 4, 5\}$ is a minimal $(2, 1)$ dominating set. But $V - D = \{1, 3\}$ is not a $(2, 1)$-dominating set. Therefore, the complement of a minimal (k, r)-dominating set need not be a (k, r)-dominating set.

Theorem 3.4 *If $r = diam(G)$ and $\left\lfloor \frac{n}{k} \right\rfloor \geq 2$, then the complement of a minimal (k, r)-dominating set is a (k, r)-dominating set.*

Proof: Let D be a minimal (k, r)-dominating set and $r = diam(G)$ and $\left\lfloor \frac{n}{k} \right\rfloor \geq 2$. **Claim:** $V - D$ is a (k, r)-dominating set. Since $r = diam(G)$, $\gamma_{(k,r)}(G) = k$. That is, $|D| = k$. Therefore $|V - D| = n - k \quad \geq k$, since $\left\lfloor \frac{n}{k} \right\rfloor \geq 2$. Since $r = diam(G)$, every vertex in $V(G)$ is at a distance $\leq r$ with every other vertex in $V(G)$. Therefore, $V - D$ is a (k, r)-dominating set.

Remark 3.5 *If H is the spanning subgraph of G, then $\gamma_{(k,r)}(G) \leq \gamma_{(k,r)}(H)$.*

Remark 3.6 *If $1 \leq s \leq r$ and $1 \leq k' \leq k$, then $\gamma_{(1,r)}(G) \leq \gamma_{(1,s)}(G) \leq \gamma_{(1,1)}(G) \leq \gamma_{(k',1)}(G) \leq \gamma_{(k,1)}(G)$.*

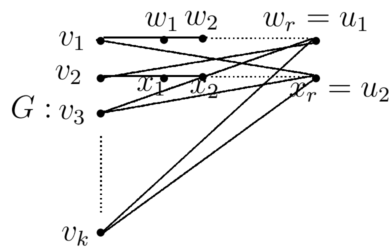
For any positive integers r and t, $\gamma_{(1,r)}(G) \leq \gamma_{(2,r)}(G) \leq \gamma_{(3,r)}(G) \leq \ldots \leq \gamma_{(t,r)}(G) \leq \gamma_{(t,r-1)}(G) \leq \ldots \leq \gamma_{(t,1)}(G) = \gamma_t(G)$, where $\gamma_t(G)$ is the t domination number of G and $\gamma_{(1,r)}(G)$ is the distance-r-domination number of G.

Proposition 3.7 *Let $n = 2t$. Then $\gamma_{(2,t-2)}(P_n) > \gamma_{(2,t-1)}(P_n) > \gamma_{(2,t)}(P_n) = \gamma_{(2,t+1)}(P_n) = \cdots = \gamma_{(2,2t)}(P_n)$.*

Proof: Let $n = 2t$. Let $V(P_n) = \left\{v_1, v_2, v_3, \ldots, v_{(2t)}\right\}$. Clearly $\{v_{(t-2)}, v_{(t-1)}, v_{(t+2)}, v_{(t+3)}\}$ is a $(2, t-2)$-dominating set of P_n. Let $\{v_i, v_j, v_k\}$ be a $(2, t-2)$-dominating set, $1 \leq i < j < k \leq 2t$. Then any two of $d(v_1, v_i)$, $d(v_1, v_j)$ and $d(v_1, v_k)$ are less than or equal to $t - 2$. That is any two of $i - 1$, $j - 1$, $k - 1$ are less than or equal to $t - 2$. Therefore, any two of i, j, k are less than or equal to $t - 1$. Let $i \leq t - 1$ and $j \leq t - 1$. Maximum value of j is $t - 1$. Then $d(v_{2t}, v_j) = 2t - j \geq 2t - (t - 1) = t + 1$. Simillarly, $d(v_{2t}, v_i) \geq t + 1$. Therefore v_{2t} is not $(2, t-2)$-dominated by v_i and v_j, a contradiction. In a similar manner, we can prove that $\{v_i, v_j, v_k\}$ is not a $(2, t-2)$-dominating set in other cases also. Therefore $\gamma_{(2,t-2)}(P_n) = 4$. Hence the remark.

Proposition 3.8 *Given positive integers k and r, there exists a connected graph G with $\gamma_{(k,r)}(G) = k$ and $diam(G) = r + 1$.*

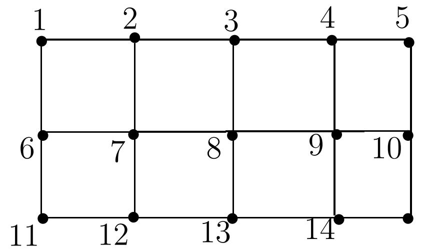
Proof: The proof is by the following construction.



Let $D = \{v_1, v_2, \ldots, v_k\}$. Let $v_1, w_1, w_2, \ldots, w_r = u_1$ be a shortest path between v_1 and u_1. Let $v_2, x_1, x_2, \ldots, x_r = u_2$ be a shortest path between v_2 and u_2. Let u_1, u_2 be adjacent to $v_3, v_4, \ldots, v_k$. Let u_1 be adjacent to v_2 and u_2 be adjacent to v_1. Now $diam(G) = r + 1$. D is a (k, r)-dominating set of G and $|D| = k$. Therefore $\gamma_{(k,r)}(G) \leq |D| = k$. But $k \leq \gamma_{(k,r)}(G)$. Therefore D is a $\gamma_{(k,r)}$-set of G and $r < diam(G) = r + 1$.

4 Vizing Conjecture:

For any graph G and H, $\gamma(G \times H) \geq \gamma(G)\gamma(H)$. But in the case of (k, r)-domination,
$\gamma_{(k,r)}(G \times H) < \gamma_{(k,r)}(G)\, \gamma_{(k,r)}(H)$, for some k and r.



In the above example, $\{1, 3, 5, 7, 9, 11, 13, 15\}$ is a $\gamma_{(3,1)}$ set of G. $\gamma_{(3,1)}(P_3 \times P_5) = 8$. $\gamma_{(3,1)}(P_3) = 3$. $\gamma_{(3,1)}(P_5) = 5$. Hence, $\gamma_{(3,1)}(P_3 \times P_5) < \gamma_{(3,1)}(P_3)\gamma_{(3,1)}(P_5)$. In the above example, $\{6, 7, 9, 10\}$ is a $\gamma_{(2,2)}$ set of G. $\gamma_{(2,2)}(P_3 \times P_5) = 4$. $\gamma_{(2,2)}(P_3) = 2$. $\gamma_{(2,2)}(P_5) = 3$. Hence $\gamma_{(2,2)}(P_3 \times P_5) < \gamma_{(2,2)}(P_3)\gamma_{(2,2)}(P_5)$.

In the above example, $\{3, 8, 13\}$ is a $\gamma_{(2,3)}$ set of G. $\gamma_{(2,3)}(P_3 \times P_5) = 3$. $\gamma_{(2,3)}(P_3) = 2$. $\gamma_{(2,3)}(P_5) = 2$. Hence $\gamma_{(2,3)}(P_3 \times P_5) < \gamma_{(2,3)}(P_3)\gamma_{(2,3)}(P_5)$.

5 Conclusion:

We have made a study of (k, r)-domination. It is further continued in our subsequent investigations in this direction. Facility location problems as considered in **[5]** make use of (k, r)-domination. Other applications are also attempted.

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Received: February, 2011