Primary Ideals in Boolean Like Semi Rings

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Abstract

In this we introduce the concept of ideal quotients and primary ideals in a Boolean like semi ring and obtain some properties these ideals. Further we give a necessary and sufficient condition for a primary ideal to be prime in a Boolean like semi ring.

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Introduction

The concept of Boolean like semi rings is due to Venkateswarlu and Murthy in [9]. This paper is a continuation of the study on Boolean like semi rings by introducing the concept of quotient ideals and primary ideals in a Boolean like semi ring. In section 1, we give preliminary concepts and results regarding Boolean like semi rings. In section 2, we
K. Venkateswarlu and B. V. N. Murthy introduce the concept of ideal quotients of two ideals and obtain that ideal quotient of two ideals I and J of a Boolean like semi ring is once again an ideal containing the ideal I (See theorem 2.2 (i) and (ii)). Further we obtain that a weak commutative Boolean semi ring free from zero divisors is a commutative ring (see theorem 2.3) and also prove that a weak commutative Boolean like semi ring R is a Boolean like ring if R is free from zero divisors having unity. Finally in section 3, we introduce the concept of primary ideals and give necessary and sufficient condition for a primary ideal to be a prime ideal in a Boolean like semi ring. Further we make a few remarks regarding certain results on primary ideals which are true Boolean like rings of Swaminathan [6] but fail in Boolean like semi rings.

1. Boolean like semi rings and its properties

We recall certain definitions and results concerning Boolean like semi rings from [9]

**Definition 1.1.** A non empty set R together with two binary operations + and . satisfying the following conditions is called a Boolean like semi ring

1. (R,+) is an abelian group
2. (R,.) is a semi group
3. a.(b+c) = a.b + a.c for all a, b, c ∈ R
4. a + a = 0 for all a in R
5. ab( a+b+ab ) = ab for all a, b ∈ R.

Let R be a Boolean like semi ring. Then

**Lemma 1.2** For a ∈ R , a.0 = 0

**Lemma 1.3** For a ∈ R , a^4 = a^2 (weak idempotent law)

**Remark 1.4.** From the above lemma 1.3, we have a^{2n} = a^2, for any integer n > 0

**Lemma 1.5.** If R is a Boolean like semi ring then, a^n = a or a^2 or a^4 for any integer n > 0

**Definition 1.6.** A Boolean like semi ring R is said to be weak commutative if abc = acb, for all a,b,c ∈ R.
Lemma 1.7. If \( R \) is a Boolean like semi ring with weak commutative then \( 0.a = 0 \) for all \( a \in R \).

Lemma 1.8. Let \( R \) be Boolean like semi ring then for any \( a, b \in R \) and for any integers \( m, n \)

\[(1) \ a^m a^n = a^{m+n} \quad (2) \ (a^m)^n = a^{mn} \quad (3) \ (ab)^n = a^n b^n \] if \( R \) is weak commutative.

Definition 1.9. A non empty subset \( I \) of \( R \) is said to be an ideal if

1. \( (I,+ ) \) is a sub group of \( (R,+ ) \), i.e., for \( a, b \in R \Rightarrow a + b \in R \)
2. \( ra \in R \) for all \( a \in I, r \in R \), i.e., \( RI \subseteq I \)
3. \( (r+a)s + rs \in I \) for all \( r, s \in R, a \in I \)

Remark 1.10. If \( R \) is weak commutative Boolean like semi ring then \( ar \in I \) for all \( a \in I \) and \( r \in R \)

Definition 1.11. If \( R \) is a weak commutative Boolean like semi ring then the radical of an ideal \( I \) denoted by \( r(I) = \{ x \in R : x^n \in I \text{ for some positive integer } n \} \)

Theorem 1.12. If \( I \) is an ideal of a weak commutative Boolean like semi ring \( R \) then

(a) \( I \subseteq r(I) \), \quad (b) \( r(I \cap J) = r(I) \cap r(J) \),
(c) \( r(I) \subseteq r(J) \) \quad (d) \( r(r(I)) = r(I) \)

Definition 1.13. The nilradical \( N(R) \) is the set of all nilpotent elements of a weak commutative Boolean like semi ring \( R \).

Theorem 1.14. \( N(R) \) forms an ideal of \( R \)

Remark 1.15. We write \( N \) instead of \( N(R) \) for convenience

2 Ideal quotients

We now introduce the concept of ideal quotients in Boolean like semi rings.

Definition 2.1. Let \( R \) be a weak commutative Boolean like semi ring. Let \( I \) and \( J \) be ideals of \( R \). Then their ideal quotient denoted by \( (I:J) \) and is defined by

\[ (I:J) = \{ x \in R / Jx \subseteq I \} \]

Now we have the following
Theorem 2.2. If \( R \) is a weak commutative Boolean like Semi ring and \( I, J \) and \( K \) are the ideals of \( R \) then the following hold.

(i) \((I:J) = \{ x \in R/ Jx \subseteq I \} \) is an ideal of \( R \).

(ii) \( I \subseteq (I:J) \)

(iii) \(((I:J):K) = (I:JK)\)

(iv) \((\cap I_i : J) = \cap_i (I_i : J)\)

Proof. (i) Clearly \((I:J)\) is non empty subset of \( R \)

Let \( x, y \in (I:J) \), then \( Jx \subseteq I \), \( Jy \subseteq I \). Hence \( ax \in I \), \( ay \in I \) for all \( a \in J \)

Consider \( a(x+y) = ax + ay \in I \), for all \( a \in J \). Hence \( J(x+y) \subseteq I \).

Hence \( x + y \in (I:J) \)

Let \( r \in R \), \( x \in (I:J) \). \( \Rightarrow \) \( Jx \subseteq I \) \( \Rightarrow \) \( ax \in I \) for \( a \in J \)

\( \Rightarrow \) \( axr \in I \) (by remark 1.10) \( \Rightarrow \) \( arx \in I \), for all \( a \in J \) (by weak commutativity)

\( \Rightarrow \) \( rax \in (I:J) \). Finally let \( r, s \in R \), \( x \in (I:J) \). Then \( Jx \subseteq I \) \( \Rightarrow \) \( ax \in I \), for all \( a \in J \).

Consider \( a[(r+x)s + rs] = a(r+x)s + ars \) (by def 1.1(3))

\[ = as(r+x) + ars \quad \text{(by weak commutativity)} \]

\[ = asr + asx + ars \quad \text{(by weak commutativity & by def 1.1(3))} \]

\[ = asx \quad \text{(by 1.1(4))} = axs \quad \text{(by weak commutativity)} \]

\[ = axs \quad \text{(by weak commutativity)} \]

Hence \( J[(r+x)s + rs] \subseteq I \) \( \Rightarrow \) \( (r+x)s + rs \in (I:J) \).

(ii) is obvious

(iii) Let \( x \in ((I:J):K) \) \( \Leftrightarrow \) \( Kx \subseteq (I:J) \) \( \Leftrightarrow \) \( J(Kx) \subseteq I \) \( \Leftrightarrow \) \( (JK)x \subseteq I \) \( \Leftrightarrow \) \( x \in (I:JK) \)

(iv) Let \( x \in (\cap I_i : J) \) \( \Leftrightarrow \) \( Jx \subseteq \cap I_i \) \( \Leftrightarrow \) \( Jx \subseteq I_i \) for all \( i = 1,2,3,\ldots,n \).

\( \Leftrightarrow \) \( x \in (I_i : J) \) for all \( i = 1,2,3,\ldots,n \) \( \Leftrightarrow \) \( x \in \cap_i (I_i : J) \)
Theorem 2.3. If $R$ is a weak commutative Boolean like Semi ring and $R$ has no zero divisors then $R$ is commutative ring.

Proof. Let $a, b \in R$ such that $a \neq 0, b \neq 0$.

Consider $aab = aba$ (by weak commutativity)

$\Rightarrow aab + aba = 0 \Rightarrow a(ab + ba) = 0 \Rightarrow ab + ba = 0$ (since $a \neq 0$) $\Rightarrow ab = ba$

Theorem 2.4. If $R$ is a weak commutative Boolean like Semi ring in which $1$ is a unity element and has no zero divisors then $R$ is a Boolean like ring.

Proof. $R$ is commutative follows from Theorem 2.3

It needs only to show $ab(1+a)(1+b) = 0$ for all $a, b \in R$

$ab(a+b+ab) = ab$ (by definition 1.1(5))

$\Rightarrow aba + abb + abab = ab$ (by def 1.1(3))

$\Rightarrow aba + ab + abab + ab = 0 \Rightarrow (aba + ab) + (abb + abab) = 0$

$\Rightarrow ab(a+1) + ab(b+ab) = 0 \Rightarrow ab\{(a+1) + (b+ba)\} = 0$

$\Rightarrow ab\{(a+1) + b(1+a)\} = 0 \Rightarrow ab(1+a)(1+b) = 0$

Theorem 2.5. If $R$ is a weak commutative Boolean like semi ring then for any ideal $I$ of $R$ $IR \subseteq I$ and hence $RIR \subseteq I$.

Proof. $(r + a)s + rs \in I$, for any $a \in I$ and $r,s \in R$. Letting $r = 0$ then $IR \subseteq I$.

Also $RI \subseteq I$, hence $RIR \subseteq IR \subseteq I$

Remark 2.6. If $R$ is a Boolean like Semi ring in which $a^2 = a$, for all $a \in R$ then $R$ is a Boolean near ring.
Example 2.7. Let \( R = \{0, x, y, z\} \). \( + \) and \( . \) are defined as follows

\[
\begin{array}{cccc}
+ & 0 & x & y & z \\
0 & 0 & x & y & z \\
x & x & 0 & z & y \\
y & y & z & 0 & x \\
z & z & y & x & 0 \\
\end{array}
\quad \begin{array}{cccc}
. & 0 & x & y & z \\
0 & 0 & 0 & 0 & 0 \\
x & 0 & x & 0 & x \\
y & 0 & 0 & y & 0 \\
z & 0 & z & 0 & z \\
\end{array}
\]

Clearly \( R \) is Boolean near ring and as well as Boolean like semi ring. It is observed in [9] that these two class of rings are independent.

Remark 2.8. If \( R \) is a weak commutative Boolean like Semi ring with unity \( 1 \) then \( R \) is a Boolean like ring.

3 Primary ideals

Definition 3.1. An ideal \( I \) of a Boolean like semi ring \( R \) is primary if \( I \neq R \) and for any \( x, y \in R \), \( xy \in I \) implies \( x \in I \) or \( y^n \in I \), for some positive integer \( n \).

Theorem 3.2. \( I \) is primary \( \iff \) \( R/I \neq \{0\} \) and every zero divisor (either left or right) in \( R/I \) is nilpotent.

Proof. Suppose \( I \) is primary. Then \( I \neq R \) and \( I \) is proper ideal of \( R \) and hence \( R/I \neq \{0\} \).

Let \( x+I \in R/I \) be a zero divisor in \( R/I \). Then there exists \( 0+I \neq y+I \in R/I \) such that
\[
(x+I)(y+I) = 0 +I \implies xy+I = 0+I \implies xy \in I \implies x^n \in I \ (\text{since}, \ y \not\in I) \implies x^n + I = 0 + I \implies (x+I)^n = 0+I. \text{ Hence } x+I \text{ is nilpotent.}
\]

Conversely, let \( xy \in I \) such that \( x \not\in I \). Then \( xy + I = 0 + I \)
\[
\implies (x+I)(y+I) = 0+I \implies y+I \text{ is a zero divisor in } R/I \implies y+I \text{ is nilpotent in } R/I
\]
\[
\implies (y+I)^n = 0+I \implies y^n + I = 0+I \implies y^n \in I. \text{ Hence } I \text{ is primary.}
Theorem 3.3. If $I$ is a primary ideal of a weak commutative Boolean like semi ring $R$, then $r(I)$ is the smallest prime ideal containing $I$.

**Proof.** Let $xy \in r(I)$ such that $x \not\in r(I)$. Then $x^n \not\in I$, for any $n > 0$. Thus $xy \in r(I)$

$\Rightarrow (xy)^m \in I$, for some $m > 0 \Rightarrow x^m y^m \in I$ (from lemma 1.8 (3))

$\Rightarrow (y^m)^i \in I$ (since $I$ is primary) $\Rightarrow y^{mi} \in I$ (from lemma 1.8(2)) $\Rightarrow y \in r(I)$.

Hence $r(I)$ is prime ideal. By theorem 1.12, $I \subseteq r(I)$ is clear.

Let $J$ be any prime ideal of $R$ containing $I$. We now prove that $r(I) \subseteq J$. Let $x \in r(I)$

then $x^n \in I$, for some positive integer $n \Rightarrow x^n \in I \subseteq J \Rightarrow x^n \in J \Rightarrow x \in J$, since $J$ is prime

$\Rightarrow r(I) \subseteq J$. Hence $r(I)$ is the smallest prime ideal of $R$ containing $I$.

Theorem 3.4. Let $I$ be an ideal of a Boolean like semi ring $R$. then

(1) $x + I$ is an idempotent in $R/I \iff$ the nilpotent element $x_N \in I$

(2) $(x+I)^2 = 0 + I$ in $R/I \iff$ the idempotent element $x_B \in I$.

**Proof.** (1) $x + I$ is idempotent in $R/I \iff (x+I)^2 = x+I \iff x^2 + I = x + I \iff x^2 + x \in I \iff x_N \in I$.

(2) $(x+I)^2 = 0 + I$ in $R/I \iff x^2 \in I \iff x_B \in I$.

**Definition [2]:** A near ring $N$ is called zero symmetric if $0^n = 0$ for all $n \in N$

**Remark A.**

(a) Every weak commutative Boolean like semi ring $R$ is zero symmetric.

(b) Prime ideal is primary in Boolean like semi ring $R$, but not conversely.

Consider the following example

**Example 3.5** Let $R = \{0, a, b, c\}$, $+$ and $\cdot$ are defined as follows
Clearly $R$ is a Boolean like semi ring. In this $\{0\}$ is primary and $\{0\} \subseteq \{0,a\}$ is maximal (Prime ideal). Since $ba \in \{0\}$ but $b \notin \{0\}$ and $a \notin \{0\}$, hence $\{0\}$ is not prime.

(c) Primary ideal is not prime even if $R$ is a weak commutative Boolean like semi ring.

Consider the following

Example 3.6. Let $R = \{0,x,y,z\}$, $+$ and $.$ are defined by

\[
\begin{array}{cccc}
+ & 0 & x & y \\
0 & 0 & x & y \\
x & x & 0 & y \\
y & y & z & 0 \\
z & z & y & x \\
\end{array}
\quad
\begin{array}{cccc}
\cdot & 0 & x & y \\
0 & 0 & 0 & 0 \\
x & 0 & x & 0 \\
y & 0 & 0 & 0 \\
z & 0 & z & 0 \\
\end{array}
\]

$\{0\}$ is primary and $\{0\} \subseteq \{0,y\}$ is prime ideal.

Since $yz \in \{0\}$, but $y \notin \{0\}$ and $z \notin \{0\}$, hence $\{0\}$ is not prime.

Now we give a necessary and sufficient condition for a primary ideal of a Boolean like semi ring to be prime in the following.

Theorem 3.7. In a Boolean like semi ring $R$, a primary ideal $I$ is prime if and only if the nil radical $N$ of $R$ is a sub set of $I$. 
Primary ideals in Boolean like semi rings

Proof. Suppose that $I$ is primary ideal such that nil radical $N \subseteq I$. Now we claim that $I$ is prime. Let $xy \in I$ such that $x \notin I$ then $y^n \in I \Rightarrow y \in I$ or $y^2 \in I$ or $y^3 \in I$.

If $y \in I$ then nothing to prove. If $y^2 \in I$ then $y = y + (y^2 + y^2)$ (def 1.1(4))

$$= (y + y^2) + y^2 \in N + I \subseteq I$$

( since $y + y^2$ is nil potent), Hence $y \in I$. If $y^3 \in I$ then

$$y = y + y^4 + y^4 = (y + y^2) + y y^3$$ (By weak idempotent law) $\in N + I \subseteq I$.

In any case $y \in I$. Hence $I$ is prime.

Conversely suppose that $I$ is prime. We now prove that $N \subseteq I$.

If $x \in N$ then $x^n = 0$ for some positive integer $n \Rightarrow x = 0$ or $x^2 = 0$ or $x^3 = 0$

$\Rightarrow x \in I$ or $x^2 \in I$ or $x^3 \in I \Rightarrow x \in I$ (I is prime).

We recall a few results concerning Boolean like rings of Swaminathan [6].

Theorem[6]. An ideal $I \neq R$ of a Boolean like ring is primary $\iff$ $R/I$ has only two idempotents.

Corollary[6] $\{0\}$ is primary $\iff$ the only idempotents of $R$ are 0,1.

Corollary[6] An ideal of a Boolean ring with unity is primary $\iff$ it is prime.

Theorem[6] An ideal $I$ of a Boolean like ring is primary $\iff$ for any idempotent $b \in R$

either $b \in I$ or $1+b \in I$.

Remark B. The above results need not be true in the case of Boolean like semi rings

Here we furnish an example where all the above theorems of Swaminathan [6] fail.

Example 3.8. Let $R = \{0, p, q, l \}$ , $+$ and $\cdot$ are defined as follows
Clearly $R$ is a Boolean like semi ring.

Now $I = \{0\}$ is primary ideal and $J = \{o, p\}$ is a prime ideal

Clearly $0, 1, q$ are idempotent elements of $R$. Further $R/I$ is isomorphic to $R$ and hence $R/I$ has three idempotents.

We observe that $\{0\}$ is not prime, since $qp = 0 \in \{0\}$, but $q \notin \{0\}$ and $p \notin \{0\}$

Also $R$ has three idempotents namely $0, 1, q$. But $q \notin \{0\}$ and $1+q = p \notin \{0\}$

**Remark C.** In a Boolean like ring every element can be written as a sum of nilpotent element and an idempotent element where as this is not true in the case of a Boolean like semi ring.

For instance, consider the example 3.5 defined above.

$c = 0 + c = b + a$ \((b^2 = b, a^2 = 0)\) are two different representations, also $a = b + c, b^2 = b, c^2 = c$.

**Remark D.**

1. In a Boolean like ring $R$, the set of all idempotents of $R$ form a Boolean sub ring with unity 1 denoted by $B$ where as this is not true in case of Boolean like Semi ring $R$. 
In the above example 3.5, let $B = \{0, b, c\}$ the set of idempotent elements and $b + c = a$ which is not in $B$.

2. $R/N$ is isomorphic to $B$ in a Boolean like ring $R$, where $N$ is the set of all nilpotent elements of $R$. However this fails in a Boolean like semi ring $R$.

Consider $N = \{0, a\}$, $B = \{0, b, c\}$, $R/N = \{N, b + N\}$ in example 3.5.

Clearly $R/N$ is not isomorphic to $B$.

References


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