Trace Inequalities for Operations of Matrix Powers

by Aid of Block Matrices

Zübeyde ULUKÖK
Department of Mathematics, Science Faculty
Selcuk University, 42003 Konya, Turkey
zulukok@selcuk.edu.tr

Ramazan TÜRKMEN
Mevlana University, Education Faculty
Primary School Mathematics Teaching, Konya, Turkey
rturkmen@mevlana.edu.tr

Abstract

In this paper, firstly, a trace inequality depending on the determinant for positive semidefinite matrices \( A \) and \( B \),
\[
n \left( \det A \det B \right)^{m/n} \leq \text{tr} \left( AB \right)^m
\]
is established, where \( m \) is any integer. Then, we give a matrix trace inequality for positive semidefinite block matrix \[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix}
\]
\[
\left( \text{tr} \left( B^p \right) \right)^2 \leq \text{tr} \left( A^p \right) \text{tr} \left( C^p \right),
\]
where \( p \) is a positive integer. The applications of the above inequality improve some results which are on traces of the Hadamard products, ordinary products and sums of positive semidefinite matrices given by some authors ([1], [3], [9], [10]).

Keywords: Positive Semidefinite Matrices, Trace Inequalities, Majorization

1. Introduction

Inequalities have proved to be a powerful tool in mathematics. Matrix
inequalities arise in various branches of mathematics and science such as system and control theory, optimization, in modeling error analysis for filtering and estimation problems, in adaptive stochastic control and semidefinite programming ([2], [5], [8]). This paper is concerned with inequalities for traces of sums, products and Hadamard products of matrices. Let us begin with some terminology and notation.

Let \( M_n \) be the space of \( n \times n \) complex matrices. \( I_n \) be the identity matrix in \( M_n \). A matrix \( A \in M_n \) is Hermitian if \( A^* = A \), where \( A^* \) denotes the conjugate transpose of \( A \). A Hermitian matrix \( A \) is said to be positive semidefinite or nonnegative definite, written as \( A \geq 0 \), if
\[
x^* Ax \geq 0, \quad \text{for all} \quad x \in \mathbb{C}^n
\] (1)
\( A \) is further called positive definite, symbolized \( A > 0 \), if the strict inequality in (1) holds for all nonzero \( x \in \mathbb{C}^n \) (see, e.g., [4], p.159). If \( A, B \) are Hermitian matrices, we write \( A \geq B \) if \( A - B \geq 0 \).

For \( A = (a_{ij}), B = (b_{ij}) \in M_n \), the Hadamard product of \( A \) and \( B \) is
\[
A \odot B = [a_{ij}b_{ij}] \in M_n.
\]
For vectors \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n \) with nonnegative components in decreasing order, if
\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, ..., n
\]
then we say that \( x \) is weakly majorized by \( y \) and denote \( x \prec_w y \). If in addition to \( x \prec_w y \), \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \) holds, then we say that \( x \) is majorized by \( y \) and denote \( x \prec y \).

Let the components of \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n \) be nonnegative. If
\[
\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i, \quad k = 1, 2, ..., n
\]
then we say that \( x \) is weakly log-majorized by \( y \) and denote \( x \prec_{\log} y \). If in addition to \( x \prec_{\log} y \), \( \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i \) holds, then we say that \( x \) is log-majorized by \( y \) (see, e.g., [11], p.18).

As is well known, \( x \prec_{\log} y \) implies \( x \prec_w y \) (see, e.g., [11], p.19).

A trace inequality for the product of two positive semidefinite matrices has given in [9]. When \( A, B \geq 0 \), the following inequality holds:
\[
\text{tr}(AB)^m \leq \left(\text{tr}(A)^{2m} \text{tr}(B)^{2m}\right)^{\frac{1}{2}}
\] (2)
Further more, the following four results were proved in [1], for \( A,B \geq 0 \)
Trace inequalities for operations of matrix powers

\[ \text{tr} (A \circ B)^\alpha \leq \text{tr} (A^\alpha \circ B^\alpha), \quad \alpha \leq 0 \text{ or } \alpha \geq 1 \] (3)
\[ \text{tr} (A \circ B)^\alpha \geq \text{tr} (A^\alpha \circ B^\alpha), \quad 0 \leq \alpha \leq 1 \] and
\[ \text{tr} (AB)^\alpha \leq \text{tr} (A^\alpha B^\alpha), \quad \text{whenever } |\alpha| \geq 1 \] (4)
\[ \text{tr} (AB)^\alpha \geq \text{tr} (A^\alpha B^\alpha), \quad \text{whenever } |\alpha| \leq 1 \]

Equality holds for some value of \( \alpha \) if and only if \( \alpha = -1, 0, 1 \) or \( AB = BA \).

Also, Dannan [3] has proved the following inequality: If \( A \succ 0 \) and \( B \succ 0 \), then
\[ n(\det A \det B)^{m/n} \leq \text{tr} (A^m B^m) \] (5)
for any positive integer \( m \).

2 Lemmas

**Lemma 2.1** [7]: Suppose that the block matrix
\[ A = \begin{pmatrix} L & X \\ X^* & M \end{pmatrix} \]
is positive semidefinite, where \( L \in M_m, \ M \in M_n \), and \( X \in M_{m,n} \). Then \( L \) and \( M \) are positive semidefinite and for all \( p > 0 \) and \( k = 1, \ldots, \min \{m, n\} \) we have
\[ \prod_{i=1}^{k} \sigma_i^p (X) \leq \prod_{i=1}^{k} \sigma_i^{p/2} (L) \sigma_i^{p/2} (M). \]

**Lemma 2.2** [6]: If \( A \in M_n \), then
\[ \prod_{i=1}^{k} |\lambda_i (A)| \leq \prod_{i=1}^{k} \sigma_i (A), \quad 1 \leq k \leq n \] (6)

If \( k = n \), then equality holds in (6).

**Lemma 2.3** (Cauchy-Schwarz inequality): Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be real numbers. Then
\[ \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right), \quad \forall a_i, b_i \in R. \]

We now firstly prove that an inequality for trace of product of two positive
semidefinite matrices by depending on determinant. Secondly, we present a trace inequality for positive semidefinite block matrices. Finally, we give some corollaries for this inequality by applying several positive semidefinite block matrices.

3 Main Results

Theorem 3.1. Let $A$ and $B$ be a $n \times n$ square positive semidefinite matrices. Then,

$$n \left( \det A \det B \right)^{m/n} \leq \text{tr} \left( AB \right)^m$$

where $m$ is any integer.

Proof: Since $A$ and $B$ are positive semidefinite matrices, there exist a unique $A^{1/2} \geq 0$ and $B^{1/2} \geq 0$. Then, we know that $B^{1/2} AB^{1/2} \geq 0$. Note that if $B^{1/2} AB^{1/2} \in M_n$ is positive semidefinite, then so are all the powers $(B^{1/2} AB^{1/2})^m$, $m=1, 2, \ldots$.

The arithmetic-geometric means inequality states that

$$\left( \prod_{i=1}^{n} x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i$$

Where $x_i$‘s are nonnegative real numbers and $n$ is a positive integer. Then, by using the arithmetic-geometric inequality for matrix $(B^{1/2} AB^{1/2})^m$, we write

$$\left( \det (B^{1/2} AB^{1/2})^m \right)^{1/n} \leq \frac{\text{tr} \left( B^{1/2} AB^{1/2} \right)^m}{n}$$

Since $\det (XY) = \det (X) \det (Y)$ and $\text{tr}(XY) = \text{tr}(YX)$ for any square matrices $X$ and $Y$, we get

$$n \left( \det A \det B \right)^{m/n} \leq \text{tr} \left( AB \right)^m. \quad (7)$$

Consequently, if (4), (5), and (7) are combined, then

$$n \left( \det A \det B \right)^{m/n} \leq \text{tr} \left( AB \right)^m \leq \text{tr} \left( A^m B^m \right). \quad \square$$

Block matrices in the form \[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix}
\] plays an important role in deriving matrix inequalities. We shall give some new trace inequalities and new proofs of trace inequalities previously obtained for matrices by performing the following Theorem to some positive semidefinite block matrices.

Theorem 3.2: Let $A, B, \text{ and } C$ be $n \times n$ square complex matrices such that
Trace inequalities for operations of matrix powers

Then,
\[
\left[ \text{tr} \left( |B|^p \right) \right]^2 \leq \text{tr} \left( A^p \right) \text{tr} \left( C^p \right)
\]

where \( p \) is a positive integer. If \( 0 \leq \text{tr} (B) \in R \), then
\[
\left[ \text{tr} (B)^p \right]^2 \leq \left[ \text{tr} \left( |B|^p \right) \right]^2 \leq \text{tr} \left( A^p \right) \text{tr} \left( C^p \right).
\]

Proof: From the definition of the singular value, we can write the following equality:
\[
\sum_{i=1}^{n} \left[ \sigma_i (B) \right]^p = \sum_{i=1}^{n} \left[ \lambda_i (BB^*)^{1/2} \right]^p = \sum_{i=1}^{n} \lambda_i \left( |B|^p \right) = \text{tr} \left( |B|^p \right)
\]  
(8)

and by Lemma 2.1 we have
\[
\sum_{i=1}^{n} \left[ \sigma_i (B) \right]^p \leq \sum_{i=1}^{n} \left[ \sigma_i (A) \right]^{p/2} \left[ \sigma_i (C) \right]^{p/2}
\]  
(9)

On the other hand, by applying Lemma 2.3 (Cauchy-Schwarz inequality) to (9) and by using positive semidefinite of \( A \) and \( C \), we have
\[
\left[ \sum_{i=1}^{n} \sigma_i \left( A^{p/2} \right) \sigma_i \left( C^{p/2} \right) \right] \leq \left[ \sum_{i=1}^{n} \sigma_i^2 \left( A^{p/2} \right) \right] \left[ \sum_{i=1}^{n} \sigma_i^2 \left( C^{p/2} \right) \right]^{1/2}
\]  
(10)

Since \( A \) and \( C \) are positive semidefinite, we write
\[
\sigma_i^2 \left( A^{p/2} \right) = \sigma_i (A^p) = \lambda_i (A^p) \quad \text{and} \quad \sigma_i^2 \left( C^{p/2} \right) = \sigma_i (C^p) = \lambda_i (C^p)
\]  
(11)

Thus, from equalities (8), (11) and inequality (10) we get
\[
\text{tr} (|B|)^p \leq \left[ \sum_{i=1}^{n} \lambda_i (A^p) \right] \left[ \sum_{i=1}^{n} \lambda_i (C^p) \right]^{1/2}
\]
\[
\left[ \text{tr} (|B|)^p \right]^2 \leq \text{tr} \left( A^p \right) \text{tr} \left( C^p \right).
\]

When \( \text{tr} B \) is real, by using Lemma 2.2, we can write
\[ tr(B)^p = \sum_{i=1}^{2n} (\lambda_i(B))^p \leq \sum_{i=1}^{2n} |\lambda_i(B)|^p \leq \sum_{i=1}^{n} [\sigma_i(B)]^p = tr(|B|^p) \]
\[
\left[ tr(B)^p \right]^2 \leq \left[ tr(|B|^p) \right]^2 \leq tr(A^p) tr(C^p). \]

**Corollary 3.3:** Let \( A \in M_n \) and \( B \in M_n \) be positive semidefinite matrices. We know that for any matrix \( X, X^*X \) is positive semidefinite. Then for \( A, B \geq 0 \) we write
\[
\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^2 & AB \\ BA & B^2 \end{pmatrix} \geq 0.
\]
Since \( 0 \leq tr(AB) = tr(B^{1/2}AB^{1/2}) \in R \), according to Theorem 3.2, we get
\[
tr(AB)^p \leq \left( tr(A)^{2p} tr(B^{2p}) \right)^{\frac{1}{2}}
\]
where \( p \) is a positive integer.
This inequality is a given inequality by X. M. Yang in [5].

**Corollary 3.4:** Let \( A \in M_n \) and \( B \in M_n \) be positive semidefinite matrices. For, \( A, B \geq 0 \) we know that
\[
\begin{pmatrix} A^{2m} & A^m B^m \\ B^m A^m & B^{2m} \end{pmatrix} \geq 0.
\]
Hence, by using Theorem 3.2, we get
\[
tr\left( A^m B^m \right)^p \leq \left( tr\left( A^{2m} \right)^p tr\left( B^{2m} \right)^p \right)^{\frac{1}{2}}.
\]
If we take \( p = 1 \), then
\[
tr\left( A^m B^m \right) \leq \left( trA^{2m} trB^{2m} \right)^{\frac{1}{2}}.
\]
Also, from inequalities (4), (12), and the above inequality, we write
\[
tr(AB)^m \leq tr\left( A^m B^m \right) \leq \left( tr\left( A^{2m} \right) tr\left( B^{2m} \right) \right)^{\frac{1}{2}}. \tag{13}
\]
Thus, the inequality given by X. M. Yang [5] has been generalized.

**Corollary 3.5:** Let \( A \in M_n \) and \( B \in M_n \) be positive semidefinite matrices. For \( A \geq 0 \), we know that
\[
\begin{pmatrix} A^{2m} & A^m A^{m+n} \\ A^{m+n} & A^2 \end{pmatrix} \geq 0.
\]
By Theorem 3.2, we have
Trace inequalities for operations of matrix powers

\[ tr\left(A^{m+n}\right)^p \leq \left\{ tr\left(A^{2m}\right)^p \right\} \frac{1}{2}. \]

for \( p = 1 \) we write

\[ tr\left(A^{m+n}\right) \leq \left\{ tr\left(A^{2m}\right)\right\} \frac{1}{2}. \]  \hspace{1cm} (14)

Since (14) is true for any \( A \geq 0 \), we let \( D = B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq 0 \). Then inequality (14) holds for \( D \). Thus

\[ tr\left(AB\right)^{m+n} \leq \left\{ tr\left(AB\right)^{2m} \right\} \frac{1}{2}. \]  \hspace{1cm} (15)

Also, by arithmetic-geometric means inequality we get

\[ tr\left(AB\right)^{m+n} \leq \left\{ tr\left(AB\right)^{2m} \right\} \frac{1}{2} \leq \frac{tr\left(AB\right)^{2m} + tr\left(AB\right)^{2n}}{2} \]  \hspace{1cm} (16)

where \( m \) and \( n \) are positive integers.

**Corollary 3.6:** Let \( A \in M_n \) and \( B \in M_n \) be positive semidefinite matrices. For positive semidefinite matrices \( A \) and \( B \)

\[ \begin{pmatrix} A^{2m} & A^m \\ A^m & I \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} I & B^m \\ B^m & B^{2m} \end{pmatrix} \geq 0. \]

From the property of Hadamard product of positive semidefinite, we have

\[ \begin{pmatrix} A^{2m} \circ I & A^m \circ B^m \\ A^m \circ B^m & I \circ B^{2m} \end{pmatrix} \geq 0. \]

By Theorem 3.2, we write

\[ tr\left(A^m \circ B^n\right)^p \leq \left\{ tr\left(A^{2m} \circ I\right)^p \right\} \frac{1}{2}. \]

Note that

\[ tr\left(A^{2m} \circ I\right) = tr\left(A^{2m}\right) \quad \text{and} \quad tr\left(B^m \circ I\right) = tr\left(B^m\right). \]

Thus, we get for \( p = 1 \)

\[ tr\left(A^m \circ B^n\right) \leq \left\{ tr\left(A^{2m}\right)tr\left(B^{2m}\right)\right\} \frac{1}{2}. \]  \hspace{1cm} (17)

where \( m \) is a positive integer.

**Corollary 3.7:** Let \( A \) and \( B \) be \( n \)-square matrices. Then
From Theorem 3.2, we have
\[
\left[ \text{tr} |A + B|^p \right]^2 \leq \text{tr} \left( I + AA^* \right)^p \text{tr} \left( I + B^*B \right)^p.
\] (18)

In particular, if \( A, B \succeq 0 \), then
\[
0 \leq \text{tr} \left( A + B \right)^p \leq \left\{ \text{tr} \left( I + A^2 \right)^p \text{tr} \left( I + B^2 \right)^p \right\}^{\frac{1}{2}}
\] (19)
where \( p \) is a positive integer.

Corollary 3.8: Let \( A \) be \( n \)–square matrix. For any matrix \( A \), we know that
\[
\begin{pmatrix}
AA^* & A \\
A^* & I
\end{pmatrix} \succeq 0.
\]

By Theorem 3.2, we write
\[
\text{tr} A^p \leq \left\{ \text{tr} \left( AA^* \right)^p \text{tr} I^p \right\}^{\frac{1}{2}}
\] (20)
If \( A \succeq 0 \), then
\[
0 \leq \left[ \text{tr} A^p \right]^2 \leq n\left( \text{tr} A^2 \right)^p
\] (21)
where \( p \) is a positive integer.

Since (21) is true for any \( A \succeq 0 \), we let \( D = B^\frac{1}{2}AB^\frac{1}{2} \). Then inequality (21) holds for \( D \). As result, we obtain
\[
0 \leq \left[ \text{tr} \left( AB \right)^p \right]^2 \leq n\left( \text{tr} AB \right)^{2p}.
\] (22)

Corollary 3.9. Let \( A \in M_n \) be positive semidefinite matrix. For \( A \succeq 0 \), we know that
\[
\begin{pmatrix}
A & I \\
I & A^{-1}
\end{pmatrix}.
\]

Also, for any matrices \( X \) and \( Y \) we write
\[
\begin{pmatrix}
XAX^* & XY^* \\
YX^* & YA^{-1}Y^*
\end{pmatrix} \succeq 0.
\]
Therefore, by Theorem 3.2, we get

\[
\left[ \text{tr}\left(\left| X Y^\ast \right|^p \right) \right]^2 \leq \text{tr}\left( X A X^\ast \right)^p \text{tr}\left( Y A^\ast Y^\ast \right)^p
\]

(23)

where \( X \) and \( Y \) are any \( n \) – square matrices.

In particular, in (23), if it is taken by \( X^\ast X = I_n \) and \( Y^\ast Y = I_n \), one obtains

\[
\left[ \text{tr}\left(\left| X Y^\ast \right|^p \right) \right]^2 \leq \text{tr} A^p \text{tr} A^{-p}.
\]

If \( X \) and \( Y \) be identity matrix, then we get

\[
n^2 \leq \text{tr} A^p \text{tr} A^{-p}.
\]

Also, since \( \left\| A^p \right\|_F = \sqrt{\text{tr} A^{2p}} \) and \( \kappa_F(A) = \left\| A^p \right\|_F \left\| A^{-p} \right\|_F \), we arrive at

\[
n \leq \kappa_F(A)
\]

where \( \kappa_F(A) \) is the Frobenius condition number of matrix \( A \) and \( m \) is a positive integer.

References


Received: October, 2010